TFY4245/FY8917 Solid State Physics, Advanced Course

Problemset 8

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SUGGESTED SOLUTION

Problem 1

(a) Insert $|1,2\rangle = \sum_{\boldsymbol{k}} a_{\boldsymbol{k}} |\boldsymbol{k}, -\boldsymbol{k}\rangle$ into $H|1,2\rangle = E|1,2\rangle$ to obtain

$$(H_0 + V_{\text{eff}} \sum_{\boldsymbol{k}} a_{\boldsymbol{k}} |\boldsymbol{k}, -\boldsymbol{k}\rangle = E \sum_{\boldsymbol{k}} |\boldsymbol{k}, -\boldsymbol{k}\rangle. \tag{1}$$

Now project this equation down on the adjoint states $\langle \boldsymbol{k}', -\boldsymbol{k}'|$ and use that $\langle \boldsymbol{k}', -\boldsymbol{k}' | \boldsymbol{k}, -\boldsymbol{k}\rangle = \delta_{\boldsymbol{k},\boldsymbol{k}'}$ using the orthonormality of these states. Using that $H_0|\boldsymbol{k}, -\boldsymbol{k}\rangle = 2\varepsilon_{\boldsymbol{k}} |\boldsymbol{k}, -\boldsymbol{k}\rangle$, we obtain the desired equation.

(b) The potential is stated to be attractive in a thin-shell $\pm\omega_0$ around the Fermi surface and zero elsewhere. Thus, we have

$$\langle \boldsymbol{k}', -\boldsymbol{k}' | V_{\text{eff}} | \boldsymbol{k}, -\boldsymbol{k}\rangle = \begin{cases} -V \text{ if } |\varepsilon_{\boldsymbol{k}} - \varepsilon_F| < \omega_0 \text{ and } |\varepsilon_{\boldsymbol{k}'} - \varepsilon_F| < \omega_0 \\ 0 \text{ otherwise} \end{cases} \tag{2}$$

Inserting this into the equation for $a_{\boldsymbol{k}}$ then gives precisely the result stated in the problem text. The first Heaviside-step function in the problem text mathematically takes care of the fact that the single particle energies $\varepsilon_{\boldsymbol{k}}$ are larger than ε_F. The two last step functions take care of the fact that the potential is zero unless it lies within the thin shell around the Fermi surface. Thus, the V in the problem text is the magnitude of the attractive interaction.

(c) We start with

$$(2\varepsilon_{\boldsymbol{k}} - E)a_{\boldsymbol{k}} = V \sum_{\boldsymbol{k}'} a_{\boldsymbol{k}'} \theta(\varepsilon_{\boldsymbol{k}'} - E_F)\theta(\omega_0 - |\varepsilon_{\boldsymbol{k}'} - E_F|)\theta(\omega_0 - |\varepsilon_{\boldsymbol{k}} - E_F|). \tag{3}$$

We can rewrite this as

$$\begin{aligned}(2\varepsilon_{\boldsymbol{k}} - E)a_{\boldsymbol{k}} &= V \int_{-\infty}^{\infty} d\varepsilon \sum_{\boldsymbol{k}'} a(\varepsilon)\delta(\varepsilon - \varepsilon_{\boldsymbol{k}'})\theta(\varepsilon - E_F)\theta(\omega_0 - |\varepsilon - E_F|)\theta(\omega_0 - |\varepsilon_{\boldsymbol{k}} - E_F|) \\ &= V \int_{-\infty}^{\infty} d\varepsilon D(\varepsilon)\theta(\varepsilon - \varepsilon_F)\theta(\omega_0 - |\varepsilon - \varepsilon_F|)\theta(\omega_0 - |\varepsilon_{\boldsymbol{k}} - E_F|). \end{aligned} \tag{4}$$

Upon renaming variables, we then obtain

$$a(\varepsilon)(2\varepsilon - E) = V \int_{\varepsilon_F}^{\varepsilon_F + \omega_0} d\varepsilon' D(\varepsilon') a(\varepsilon')\theta(\omega_0 - |\varepsilon - E_F|). \tag{5}$$

(d) To satisfy the above equation, $a(\varepsilon)$ *must* have the form

$$a(\varepsilon) = \frac{C}{2\varepsilon - E}\theta(\omega_0 - |\varepsilon_{\boldsymbol{k}} - E_F|) \tag{6}$$

where C is an ε-independent constant determined by the integral over ε'.

(e) Inserting the form of $a(\varepsilon)$ given in (d) into Eq. (5) gives a self-consistent equation for the eigenvalue E:

$$1 = V\int_{\varepsilon_F}^{\varepsilon_F+\omega_0} \frac{D(\varepsilon')d\varepsilon'}{2\varepsilon - E}. \tag{7}$$

Assuming that we can replace $D(\varepsilon)$ with $D(\varepsilon_F)$, as stated in the problem text, and introducing $\lambda = VD(\varepsilon_F)$, we obtain

$$1 = \lambda\int_{\varepsilon_F}^{\varepsilon_F+\omega_0} \frac{d\varepsilon'}{2\varepsilon - E}. \tag{8}$$

This can be rearranged to

$$\frac{1}{\lambda} = \ln\left[1 + \frac{2\omega_0}{\Delta}\right] \tag{9}$$

where $\Delta = 2\varepsilon_F - E$. Note that since V is a positive quantity, λ is positive, and hence Δ is required for the solution of the above equation. In effect, $E < 2E_F$.

(f) We have

$$\Delta = \frac{2\omega_0}{\mathrm{e}^{1/\lambda} - 1} \simeq 2\omega_0 e{-}1/\lambda. \tag{10}$$

for $\lambda \ll 1$. This reveals why a perturbation theory could not have given sensible result for this effect. Expanding the exponential gives

$$\Delta = 2\omega_0(1 - 1/\lambda + 1/\lambda^2 - \ldots) \tag{11}$$

For small λ, this means that each term becomes progressively larger, and the series cannot be truncated. Thus, we could not have hoped to obtain this result to any finite order in perturbation theory.

Problem 2

(a) Inserting our wavefunctions $\Psi_j = \rho_j^{1/2}\mathrm{e}^{i\theta_j}$ into the Schrödinger equation for Ψ_L and Ψ_R and equate real and imaginary parts, we obtain four equations

$$\begin{aligned}
\frac{\partial\phi_r}{\partial t} &= -\frac{K}{\hbar}\left(\frac{\rho_l}{\rho_r}\right)^{1/2}\cos\gamma,\\
\frac{\partial\phi_l}{\partial t} &= -\frac{K}{\hbar}\left(\frac{\rho_r}{\rho_l}\right)^{1/2}\cos\gamma,\\
\frac{\partial\rho_r}{\partial t} &= -\frac{2K}{\hbar}(\rho_l\rho_r)^{1/2}\sin\gamma,\\
\frac{\partial\rho_l}{\partial t} &= \frac{2K}{\hbar}(\rho_l\rho_r)^{1/2}\sin\gamma.
\end{aligned} \tag{12}$$

Here, we defined $\gamma = \theta_r - \theta_l$.

(b) Using the differential equation for $\partial\rho_L/dt$ above, we get:

$$\begin{aligned}
J &= 2e\frac{2K}{\hbar}(\rho_l\rho_r)^{1/2}\sin\gamma\\
&= J_0\sin\gamma
\end{aligned} \tag{13}$$

with $J_0 = 4K(\rho_l\rho_r)^{1/2}/\hbar$.