TFY4245/FY8917 Solid State Physics, Advanced Course

Problemset 4

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SUGGESTED SOLUTION

Problem 1

(a) We have

$$U(T) = \int_0^\infty dE\, E\, N(E) f(E)$$
$$\simeq \int_0^\mu dE\, E\, N(E) + \frac{\pi^2}{6}(k_B T)^2[\mu N'(\mu) + N(\mu)] \tag{1}$$

by using the Sommerfeld expansion given in the lecture notes, valid for temperatures far below the Fermi temperature. Subsequently, we use that

$$\int_0^\mu dE\, E\, N(E) \simeq \int_0^{\varepsilon_F} dE\, E\, N(E) + \varepsilon_F(\mu - \varepsilon_F)N(\varepsilon_F) \tag{2}$$

since ε_F and μ are close in magnitude at the low temperatures we are considering, to obtain

$$U(T) \simeq \int_0^{\varepsilon_F} dE\, E\, N(E) + \varepsilon_F\Big[(\mu - \varepsilon_F)N(\varepsilon_F) + \frac{\pi^2}{6}(k_B T)^2 N'(\varepsilon_F)\Big] + \frac{\pi^2}{6}(k_B T)^2 N(\varepsilon_F). \tag{3}$$

But from the result we found in the lecture for the temperature-dependence of the chemical potential $\mu(T)$, we see that the term inside the square bracket [...] equals zero. Therefore, we arrive at

$$U(T) = U_0 + \frac{\pi^2}{6}(k_B T)^2 N(\varepsilon_F). \tag{4}$$

(b) The free energy $F = U - TS$ contains the contribution from the entropy S of the system. The ground-state of the system at a finite temperature T is obtained by minimizing the free energy, and not the internal energy. Only at zero temperature are the internal and free energy equals.

Problem 2

(a) The state $|\Psi_0\rangle$ describes a system where we have filled each available quantum level $\boldsymbol{k}$ with two fermions of spin $s = \pm 1/2$ all the way up to the Fermi momentum $|\boldsymbol{k}| = k_F$.

(b) To prove this equation, we use

$$\hat{\psi}_s(\boldsymbol{r}) = \frac{1}{\sqrt{\Omega}} \sum_{\boldsymbol{k}} c_{\boldsymbol{k},s} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \tag{5}$$

to find that

$$\langle\psi_0|\hat{\psi}_s^\dagger(\boldsymbol{r})\hat{\psi}_{s'}^\dagger(\boldsymbol{r}')\hat{\psi}_{s'}(\boldsymbol{r}')\hat{\psi}_s(\boldsymbol{r})|\psi_0\rangle = \frac{1}{\Omega^2}\sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{q},\boldsymbol{q}'} e^{-i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}} e^{-i(\boldsymbol{q}-\boldsymbol{q}')\cdot\boldsymbol{r}'} \langle\psi_0|c_{\boldsymbol{k}s}^\dagger c_{\boldsymbol{q}s'}^\dagger c_{\boldsymbol{q}'s'} c_{\boldsymbol{k}'s}|\psi_0\rangle. \tag{6}$$

We can now distinguish between two cases. First consider the case with $s \neq s'$:

$$\langle\psi_0|c^\dagger_{\boldsymbol{k}s}c^\dagger_{\boldsymbol{q}s'}c_{\boldsymbol{q}'s'}c_{\boldsymbol{k}'s}|\psi_0\rangle = \delta_{\boldsymbol{k}\boldsymbol{k}'}\delta_{\boldsymbol{q}\boldsymbol{q}'}n_{\boldsymbol{k},s}n_{\boldsymbol{q},s'} \tag{7}$$

leading to

$$\langle\psi_0|\hat{\psi}^\dagger_s(\boldsymbol{r})\hat{\psi}^\dagger_{s'}(\boldsymbol{r}')\hat{\psi}_{s'}(\boldsymbol{r}')\hat{\psi}_s(\boldsymbol{r})|\psi_0\rangle = \frac{1}{\Omega^2}\sum_{\boldsymbol{k},\boldsymbol{q}} n_{\boldsymbol{k},s}n_{\boldsymbol{q},s'} = \frac{n^2}{4}. \tag{8}$$

Here, we used that $n = \frac{1}{\Omega}\sum_{\boldsymbol{k},s} n_{\boldsymbol{k},s}$ and that $n_{\boldsymbol{k},\uparrow} = n_{\boldsymbol{k},\downarrow}$ for a spin-degenerate normal metal described by $|\psi_0\rangle$. Note that $n = N/V$, the number of electrons per volume, while $n_{\boldsymbol{k},s}$ is the number of electrons with momentum k and spin s. Hence, the $1/\Omega^2$ factor in front of the summation above is needed to convert the total number of electrons for a given spin, $N/2 = \sum_{\boldsymbol{k}} n_{\boldsymbol{k},s}$, to a density. Secondly, assume $s = s'$ such that

$$\langle\psi_0|c^\dagger_{\boldsymbol{k}s}c^\dagger_{\boldsymbol{q}s'}c_{\boldsymbol{q}'s'}c_{\boldsymbol{k}'s}|\psi_0\rangle = (\delta_{\boldsymbol{k}\boldsymbol{k}'}\delta_{\boldsymbol{q}\boldsymbol{q}'} - \delta_{\boldsymbol{k}\boldsymbol{q}'}\delta_{\boldsymbol{q}\boldsymbol{k}'})n_{\boldsymbol{k},s}n_{\boldsymbol{q},s'} \tag{9}$$

which follows from using the anticommutation relations of the $c_{\boldsymbol{k},s}$-operators.

Adding now these two cases gives us the desired n^2 term, and what remains to show is that

$$\frac{1}{\Omega^2}\sum_s\sum_{\boldsymbol{k},\boldsymbol{q}} \mathrm{e}^{i(\boldsymbol{q}-\boldsymbol{k})\cdot(\boldsymbol{r}-\boldsymbol{r}')} n_{\boldsymbol{k},s}n_{\boldsymbol{q},s} = G(\boldsymbol{r}-\boldsymbol{r}') \tag{10}$$

We find, defining $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$, that:

$$\begin{aligned}
\frac{1}{\Omega^2}\sum_s\sum_{\boldsymbol{k},\boldsymbol{q}} \mathrm{e}^{i(\boldsymbol{q}-\boldsymbol{k})\cdot\boldsymbol{R}} n_{\boldsymbol{k},s}n_{\boldsymbol{q},s} &= \frac{2}{\Omega^2}\Big(\sum_{\boldsymbol{q}} \mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{R}} n_{\boldsymbol{q},s}\Big)\Big(\sum_{\boldsymbol{k}} \mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{R}} n_{\boldsymbol{k},s}\Big) \\
&= \frac{2}{\Omega^2}\Big|\sum_{\boldsymbol{q}} \mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{r}} n_{\boldsymbol{k},s}\Big|^2 \\
&= 2\Big|\int_{|\boldsymbol{k}|\le k_F} \frac{d^3k}{(2\pi)^3}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{R}}\Big|^2 \\
&= 2\Big(\frac{1}{2\pi^2 R}\int_0^{k_F}\int dk\, k\, \sin(kr)\Big)^2 \\
&= 2\Big(\frac{1}{2\pi^2}\frac{\sin(k_F R) - k_F R\cos(k_F R)}{R^3}\Big)^2
\end{aligned} \tag{11}$$

where $k = |\boldsymbol{k}|$ and $R = |\boldsymbol{R}|$. By finally using that $n = k_F^3/(3\pi^2)$, we arrive at the equation we were supposed to prove.