TFY4245/FY8917 Solid State Physics, Advanced Course

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Problemset 9

Problem 1

Consider free electrons in 3D moving in the presence of a uniform magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$. The eigenenergies are

$$E_{n,k_x,k_y,k_z} = \hbar\omega_c(n+1/2) + \hbar^2 k_z^2/2m. \tag{1}$$

(a) Derive the expression for density of states $D(\varepsilon)$ (per spin) written down in the lectures

$$D(\varepsilon) = \frac{1}{\pi\hbar}\sqrt{\frac{m}{2}}\frac{N_L}{L^2}\sum_{n=0}^{\infty}\frac{\Theta(\varepsilon - [n+1/2]\hbar\omega_c)}{\sqrt{\varepsilon - [n+1/2]\hbar\omega_c}} \tag{2}$$

by using that the density of states is a sum of the density of states for a 1D electron gas shifted to the minimum energies $\hbar\omega_c(n+1/2)$, according to the above expression for the eigenenergies.

We now want to compute the total zero-temperature energy of the system, E_{tot}, which enables determination of *e.g.* the magnetic susceptibility. Define the following functions

$$P_1(x) \equiv \int_0^x d\varepsilon D(\varepsilon),$$
$$P_2(x) \equiv \int_0^x d\varepsilon P_1(x). \tag{3}$$

(b) Write down an expression for the total energy of the system expressed as an integral over $D(\varepsilon)$ and perform this integral to show that

$$E = \mu N - 2VP_2(\mu). \tag{4}$$

Assuming the experimentally relevant regime $\hbar\omega_c \ll \mu$, the sum over n in $P_2(\mu)$ can be replaced by an integral. Use the Poisson summation formula

$$\sum_{n=0}^{\infty} f(n+1/2) = \int_0^{\infty} dx f(x) + 2\sum_{s=0}^{\infty}(-1)^s \int_0^{\infty} dx f(x)\cos(2\pi s x) \tag{5}$$

and

$$\sum_{s=1}^{\infty}\frac{(-1)^s}{s^2} = -\frac{\pi^2}{12} \tag{6}$$

to show that

$$P_2 \simeq \frac{1}{\pi\hbar}\sqrt{\frac{m}{2}}\frac{N_L}{L^2}\frac{4}{3}\left[\frac{2}{5}\frac{\mu^{5/2}}{\hbar\omega_c} - \frac{1}{10\sqrt{2}}(\hbar\omega_c)^{3/2} - \frac{1}{16}\hbar\omega_c\sqrt{\mu}\right] \tag{7}$$

when you neglect terms in P_2 which are rapidly oscillating in the limit $\mu/\hbar \to \infty$, corresponding to the low-field limit $B \to 0$.

(c) Write out the resulting expression for E to lowest order in B to show that

$$E = E(B=0) + \kappa B^2 \tag{8}$$

and identify an expression for κ.

Problem 2

When expressing operators in real space in terms of a Fourier-series over momentum-space operators, we frequently encounter summations over momentum and/or lattice coordinate. We have seen this both for fermions in a tight-binding model and for magnons in a Heisenberg model.

Prove that

$$\sum_k \mathrm{e}^{ik(r_j - r_{j'})} = N\delta_{jj''} \tag{9}$$

when the components of the wavevectors take values $2\pi m/L$ where m are integers that take values $-N/2, \ldots N/2-1$ so that the wavevector lies in the 1st BZ $[-\pi/a, \pi/a\rangle$ and $j = 1, 2, \ldots N$ for a 1D system. The above can be straightforwardly generalized to 2D and 3D.