TFY4245/FY8917 Solid State Physics, Advanced Course

Problemset 8

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Problem 1

In this problem, we will consider what happens with the filled Fermi-sea of a metal if we allow two electrons to interact attractively, even if the interaction is very weak in magnitude.

Consider a filled Fermi-sea at zero temperature. The highest energy state filled is at the Fermi level E_F. Now add two extra electrons (necessarily above the Fermi level) which interact attractively. For concreteness, assume that this interaction is such that it only exists if the energy of the electrons lies within a small window of energy $\pm\omega_0$ from the Fermi-surface and if the momenta of the electrons are on opposite sides of the Fermi-surface, in which case the interaction potential is a constant. Otherwise, the interaction is zero. Denote the two-particle state of two extra electrons in the absence of interactions $|\boldsymbol{k},-\boldsymbol{k}\rangle$ and in the presence of interactions $|1,2\rangle$. The total Hamiltonian is

$$H = H_0 + V_{\text{eff}} \tag{1}$$

where

$$H_0|\boldsymbol{k},-\boldsymbol{k}\rangle = 2\varepsilon_{\boldsymbol{k}}|\boldsymbol{k},-\boldsymbol{k}\rangle,\ H|1,2\rangle = E|1,2\rangle \tag{2}$$

and $\varepsilon_{\boldsymbol{k}}$ is the single-particle excitation energy of non-interacting fermions.

(a) Assume that the state $\{|\boldsymbol{k},-\boldsymbol{k}\rangle\}$ form a complete, orthonormal set of states so that we may expand the two-particle eigenstate in the presence of interactions as

$$|1,2\rangle = \sum_{\boldsymbol{k}} a_{\boldsymbol{k}}|\boldsymbol{k},-\boldsymbol{k}\rangle. \tag{3}$$

Show how the following equation for the coefficients $a_{\boldsymbol{k}}$ may be derived:

$$(2\varepsilon_{\boldsymbol{k}} - E)a_{\boldsymbol{k}} = -\sum_{\boldsymbol{k}'} a_{\boldsymbol{k}'}\langle\boldsymbol{k}',-\boldsymbol{k}'|V_{\text{eff}}|\boldsymbol{k},-\boldsymbol{k}\rangle. \tag{4}$$

(b) Consider the description of V_{eff} in the initial text of the problem. Use this to write down an explicit expression for the matrix element $\langle\boldsymbol{k}',-\boldsymbol{k}'|V_{\text{eff}}|\boldsymbol{k},-\boldsymbol{k}\rangle$. Then, show that the equation for $a_{\boldsymbol{k}}$ takes the form

$$(2\varepsilon_{\boldsymbol{k}} - E)a_{\boldsymbol{k}} = V\sum_{\boldsymbol{k}'} a_{\boldsymbol{k}'}\theta(\varepsilon_{\boldsymbol{k}} - E_F)\theta(\omega_0 - |\varepsilon_{\boldsymbol{k}} - E_F|). \tag{5}$$

What is V?

(c) First, note that $a_{\boldsymbol{k}}$ only depends on $\boldsymbol{k}$ via $\varepsilon_{\boldsymbol{k}}$. Then, show that by introducing the general definition of the density of states $D(\varepsilon) = \sum_{\boldsymbol{k}}\delta(\varepsilon - \varepsilon_{\boldsymbol{k}})$, one arrives at

$$a(\varepsilon)(2\varepsilon - E) = V\int_{E_F}^{E_F+\omega_0} d\varepsilon' D(\varepsilon')a(\varepsilon'). \tag{6}$$

(d) From the expression in (c), identify precisely how $a(\varepsilon)$ must depend on ε.

(e) Using the result for $a(\varepsilon)$ you found in (d), show that one obtains the following equation which determines the eigenvalue E of the interacting fermions:

$$1 = V\int_{E_F}^{E_F+\omega_0} \frac{D(\varepsilon')d\varepsilon'}{2\varepsilon' - E}. \tag{7}$$

Then, solve this equation and find an analytical expression for E by assuming that $D(\varepsilon)$ varies slowly near the Fermi level and can be replaced by its value $D(\varepsilon = E_F)$.

From the expression you find for E, you will see that $E < 2E_F$. In other words, an arbitrarily weak attractive interaction leads to an exact two-particle state which has a total energy that is lower than twice the Fermi energy. This indicates that the Fermi-sea would suffer some sort of collapse so that the states below the top of the Fermi-sea become accessible. If this is the case, we would expect a qualitatively major transformation of the nature of the system from a metal to something else (which turns out to be a superconductor).

(f) Above, you solved this problem exactly. Assume now $\lambda \ll 1$ and express Δ in terms of ω_0 and λ. Based on this expression for Δ, do you expect that you could have solved this problem using perturbation theory?

Problem 2

A supercurrent can flow not only inside a superconductor, but actually even through a non-superconducting material that is sandwiched between two superconductors. This is called the Josephson effect.

Let us give a simple derivation of the Josephson effect. The starting point is to treat the superconductivity via its order parameter $\Psi = |\Psi|e^{i\theta}$ as we did in Ginzburg-Landau theory in the lectures. Assume that this can be treated as a "wavefunction" for the Cooper pairs in the superconductor. Then, $|\Psi| = \rho^{\frac{1}{2}}$ where ρ is the density of Cooper pairs and θ is the macroscopic superconducting phase. If we have two superconductors L and R and electrons can move between these regions via tunneling, we can accont for this in the time-dependent Schrödinger equation describing $\Psi_{L,R}$. Assume that a single superconductor is described by

$$i\hbar\frac{\partial\Psi_j}{\partial t} = E\Psi_j \tag{8}$$

where $j = L,R$. Here, E is the ground-state energy. Let us set this energy to zero as a reference level, so that we measure all energies from the ground-state. To account for the presence of a second superconductor due to tunneling of electrons across some non-superconducting region, we include an additional term in the Hamiltonian which depends on the superconducting wavefunction in that region:

$$i\hbar\frac{\partial\Psi_L}{\partial t} = K\Psi_R,\ i\hbar\frac{\partial\Psi_R}{\partial t} = K\Psi_L. \tag{9}$$

We allow for both the densities and phases to be different in the superconductors, i.e. $\Psi_j = \rho_j^{1/2}e^{i\theta_j}$.

(a) Use the above Schrödinger equations to find an expression for $\partial\rho_L/\partial t$.

(b) The supercurrent J carried by Cooper pairs flowing in the junction (from left to right) should be

$$J = 2e\frac{\partial\rho_L}{\partial t} \tag{10}$$

since a current flow means that Cooper pairs are leaking out of the superconductor and thus changing the density ρ_L. If ρ_L is shrinking, it means Cooper pairs are moving from left to right. Each pair carries charge $-2e$. This should give rise to a negative current flowing from left to right (since a positive current is defined by the direction in which positive charges flow), consistent with the above equation. Show that this gives

$$J = J_0 \sin(\Delta\theta) \tag{11}$$

and identify an analytical expression for J_0 and $\Delta\theta$. This results tells us that when a phase difference exists between two superconductors, a supercurrent can flow between them (even through a non-superconducting material in between the superconductors, so long that this material is short enough to not yield $K \to 0$).