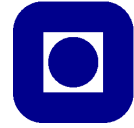


FY3464 Quantum Field Theory

Problemset 10

NTNU



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Problem 1

Using spin-wave theory, we derived in the lectures the following expression for the spin-wave induced correction ΔM to the magnetization M in a ferromagnet:

$$\Delta M = \frac{1}{N} \sum_{\mathbf{k}} \frac{1}{e^{J|S\mathbf{k}^2/k_B T} - 1}. \quad (1)$$

Evaluate this sum by turning it into an integral for a 3D system and find the explicit temperature dependence of ΔM as $T \rightarrow 0$.

Problem 2

We derived in the lectures the following Hamiltonian for an antiferromagnet with magnons:

$$H = -NJS^2z/2 + JSz \sum_{\mathbf{k}} [\gamma_{\mathbf{k}} (a_{\mathbf{k}} b_{-\mathbf{k}} + a_{\mathbf{k}}^\dagger b_{-\mathbf{k}}^\dagger) + a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + b_{\mathbf{k}}^\dagger b_{\mathbf{k}}]. \quad (2)$$

Here, the \mathbf{k} -sum is over the reduced BZ and $\gamma_{\mathbf{k}} = \frac{2}{z} \sum_{\delta} \cos(\mathbf{k} \cdot \delta)$ where z is the number of nearest neighbors and δ is the nearest-neighbor vector.

Diagonalize this Hamiltonian in the following way.

(a) Introduce a Bogoliubov transformation and introduce a new set of bosonic operators $\alpha_{\mathbf{k}}, \beta_{\mathbf{k}}$ as follows:

$$\begin{aligned} \alpha_{\mathbf{k}} &= u_{\mathbf{k}} a_{\mathbf{k}} - v_{\mathbf{k}} b_{-\mathbf{k}}^\dagger, \\ \beta_{\mathbf{k}} &= u_{\mathbf{k}} b_{\mathbf{k}} - v_{\mathbf{k}} a_{-\mathbf{k}}^\dagger. \end{aligned} \quad (3)$$

Here, $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ are two real functions of \mathbf{k} which are to be determined. Which condition must $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ fulfil in order for $\alpha_{\mathbf{k}}$ and $\beta_{\mathbf{k}}$ to satisfy bosonic commutation relations, individually?

(b) Assuming that $u_{\mathbf{k}} = u_{-\mathbf{k}}$ and $v_{\mathbf{k}} = v_{-\mathbf{k}}$ (we will check that this assumption holds in the end), the requirement $[\alpha_{\mathbf{k}}, \beta_{\mathbf{k}'}] = 0$ also holds. Find the inverse transformation of Eq. (2) and express $a_{\mathbf{k}}, b_{\mathbf{k}}$ in terms of the new boson operators.

(c) We will now choose $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ such that the Hamiltonian expressed in terms of $\alpha_{\mathbf{k}}, \beta_{\mathbf{k}}$ becomes diagonal. Show that this leads to the requirement:

$$\gamma_{\mathbf{k}} (u_{\mathbf{k}}^2 + v_{\mathbf{k}}^2) + 2u_{\mathbf{k}} v_{\mathbf{k}} = 0. \quad (4)$$

(d) Show that by parametrizing $u_{\mathbf{k}} = \cosh \theta_{\mathbf{k}}$ and $v_{\mathbf{k}} = \sinh \theta_{\mathbf{k}}$, the required condition in (a) and (c) are both fulfilled if $\theta_{\mathbf{k}}$ is chosen as

$$\tanh 2\theta_{\mathbf{k}} = -\gamma_{\mathbf{k}}. \quad (5)$$

(e) Show that since $\gamma_{-\mathbf{k}} = \gamma_{\mathbf{k}}$, the quantities $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ are indeed even functions of \mathbf{k} , as assumed.

(f) Show that the final Hamiltonian becomes a free two-species boson gas in terms of the new boson operators:

$$H = E_0 + \sum_{\mathbf{k}} \omega_{\mathbf{k}} (\alpha_{\mathbf{k}}^\dagger \alpha_{\mathbf{k}} + \beta_{\mathbf{k}}^\dagger \beta_{\mathbf{k}}) \quad (6)$$

and identify an expression for both E_0 and the dispersion relation $\omega_{\mathbf{k}}$.

Problem 3

We showed in the lectures that the quantum correction to the magnetization for sublattice ΔM_A in a square lattice antiferromagnet could be expressed with the original magnon operators as:

$$\Delta M_A = \frac{1}{N_A} \sum_{\mathbf{k}} \langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \rangle. \quad (7)$$

The expectation value is both thermal and quantum mechanical, so that:

$$\langle O \rangle = Z^{-1} \sum_m \langle m | O | m \rangle e^{-\beta E_m} \quad (8)$$

for an operator O where $Z = \sum_m e^{-\beta E_m}$ is the partition function. Here, the sum \sum_m is over the eigenstates $\{|m\rangle\}$ of H with energy eigenvalues E_m and $\beta = 1/k_B T$.

Use the diagonalized Hamilton-operator from Problem 2 above to evaluate this expectation value and prove that

$$\Delta M_A = -\frac{1}{2} + \frac{2}{N_A} \sum_{\mathbf{k}} \left(n_{\mathbf{k}} + \frac{1}{2} \right) \frac{1}{\sqrt{1 - \gamma_{\mathbf{k}}^2}}. \quad (9)$$