NTNU
Institutt for fysikk

FY3464 Quantum Field Theory
Problemset 10

Problem 1
Using spin-wave theory, we derived in the lectures the following expression for the spin-wave induced correction ΔM to the magnetization M in a ferromagnet:

$$\Delta M = \frac{1}{N}\sum_{\boldsymbol{k}} \frac{1}{\mathrm{e}^{|J|Sk^2/k_BT} - 1}. \tag{1}$$

Evaluate this sum by turning it into an integral for a 3D system and find the explicit temperature dependence of ΔM as $T \to 0$.

Problem 2
We derived in the lectures the following Hamiltonian for an antiferromagnet with magnons:

$$H = -NJS^2z/2 + JSz\sum_{\boldsymbol{k}}[\gamma_{\boldsymbol{k}}(a_{\boldsymbol{k}}b_{-\boldsymbol{k}} + a^\dagger_{\boldsymbol{k}}b^\dagger_{-\boldsymbol{k}}) + a^\dagger_{\boldsymbol{k}}a_{\boldsymbol{k}} + b^\dagger_{\boldsymbol{k}}b_{\boldsymbol{k}}]. \tag{2}$$

Here, the $\boldsymbol{k}$-sum is over the reduced BZ and $\gamma_{\boldsymbol{k}} = \frac{2}{z}\sum_{\boldsymbol{\delta}}\cos(\boldsymbol{k}\cdot\boldsymbol{\delta})$ where z is the number of nearest neighbors and $\boldsymbol{\delta}$ is the nearest-neighbor vector.

Diagonalize this Hamiltonian in the following way.

(a) Introduce a Bogoliubov transformation and introduce a new set of bosonic operators $\alpha_{\boldsymbol{k}}, \beta_{\boldsymbol{k}}$ as follows:

$$\begin{aligned}\alpha_{\boldsymbol{k}} &= u_{\boldsymbol{k}}a_{\boldsymbol{k}} - v_{\boldsymbol{k}}b^\dagger_{-\boldsymbol{k}},\\ \beta_{\boldsymbol{k}} &= u_{\boldsymbol{k}}b_{\boldsymbol{k}} - v_{\boldsymbol{k}}a^\dagger_{-\boldsymbol{k}}.\end{aligned} \tag{3}$$

Here, $u_{\boldsymbol{k}}$ and $v_{\boldsymbol{k}}$ are two real functions of $\boldsymbol{k}$ which are to be determined. Which condition must $u_{\boldsymbol{k}}$ and $v_{\boldsymbol{k}}$ fulfil in order for $\alpha_{\boldsymbol{k}}$ and $\beta_{\boldsymbol{k}}$ to satisfy bosonic commutation relations, individually?

(b) Assuming that $u_{\boldsymbol{k}} = u_{-\boldsymbol{k}}$ and $v_{\boldsymbol{k}} = v_{-\boldsymbol{k}}$ (we will check that this assumption holds in the end), the requirement $[\alpha_{\boldsymbol{k}}, \beta_{\boldsymbol{k}'}] = 0$ also holds. Find the inverse transformation of Eq. (2) and express $a_{\boldsymbol{k}}, b_{\boldsymbol{k}}$ in terms of the new boson operators.

(c) We will now choose $u_{\boldsymbol{k}}$ and $v_{\boldsymbol{k}}$ such that the Hamiltonian expressed in terms of $\alpha_{\boldsymbol{k}}, \beta_{\boldsymbol{k}}$ becomes diagonal. Show that this leads to the requirement:

$$\gamma_{\boldsymbol{k}}(u^2_{\boldsymbol{k}} + v^2_{\boldsymbol{k}}) + 2u_{\boldsymbol{k}}v_{\boldsymbol{k}} = 0. \tag{4}$$

(d) Show that by parametrizing $u_{\boldsymbol{k}} = \cosh\theta_{\boldsymbol{k}}$ and $v_{\boldsymbol{k}} = \sinh\theta_{\boldsymbol{k}}$, the required condition in (a) and (c) are both fulfilled if $\theta_{\boldsymbol{k}}$ is chosen as

$$\tanh 2\theta_{\boldsymbol{k}} = -\gamma_{\boldsymbol{k}}. \tag{5}$$

(e) Show that since $\gamma_{-\boldsymbol{k}} = \gamma_{\boldsymbol{k}}$, the quantities $u_{\boldsymbol{k}}$ and $v_{\boldsymbol{k}}$ are indeed even functions of $\boldsymbol{k}$, as assumed.

(f) Show that the final Hamiltonian becomes a free two-species boson gas in terms of the new boson operators:

$$H = E_0 + \sum_{\boldsymbol{k}} \omega_{\boldsymbol{k}} (\alpha^\dagger_{\boldsymbol{k}} \alpha_{\boldsymbol{k}} + \beta^\dagger_{\boldsymbol{k}} \beta_{\boldsymbol{k}}) \tag{6}$$

and identify an expression for both E_0 and the dispersion relation $\omega_{\boldsymbol{k}}$.

Problem 3

We showed in the lectures that the quantum correction to the magnetization for sublattice ΔM_A in a square lattice antiferromagnet could be expressed with the original magnon operators as:

$$\Delta M_A = \frac{1}{N_A} \sum_{\boldsymbol{k}} \langle a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}} \rangle. \tag{7}$$

The expectation value is both thermal and quantum mechanical, so that:

$$\langle O \rangle = Z^{-1} \sum_m \langle m|O|m\rangle \mathrm{e}^{-\beta E_m} \tag{8}$$

for an operator O where $Z = \sum_m \mathrm{e}^{-\beta E_m}$ is the partition function. Here, the sum $\sum_m$ is over the eigenstates $\{|m\rangle\}$ of H with energy eigenvalues E_m and $\beta = 1/k_B T$.

Use the diagonalized Hamilton-operator from Problem 2 above to evaluate this expectation value and prove that

$$\Delta M_A = -\frac{1}{2} + \frac{2}{N_A} \sum_{\boldsymbol{k}} \left(n_{\boldsymbol{k}} + \frac{1}{2} \right) \frac{1}{\sqrt{1-\gamma^2_{\boldsymbol{k}}}}. \tag{9}$$