TFY 4205 Quantum Mechanics II

Final exam Dec. 10, 2015

Solution Set.

Problem 1

a) $\underline{\underline{[a, a^\dagger]}} = \left[\sqrt{\frac{m\omega}{2\hbar}}\, q + \frac{i}{\sqrt{2m\hbar\omega}}\, p,\ \sqrt{\frac{m\omega}{2\hbar}}\, q - \frac{i}{\sqrt{2m\hbar\omega}}\, p\right]$

$$= -i\frac{1}{2\hbar}[q,p] + i\frac{1}{2\hbar}[p,q]$$

$$= -i\frac{1}{2\hbar}(+i\hbar) + i\frac{1}{2\hbar}(-i\hbar) = \underline{\underline{1}}$$

b) From (3) and (4)

$$\left.\begin{aligned} q &= \sqrt{\frac{\hbar}{2m\omega}}\,(a^\dagger + a) \\ p &= i\sqrt{\frac{m\hbar\omega}{2}}\,(a^\dagger - a) \end{aligned}\right\}$$

$$\underline{\underline{H}} = \frac{1}{2m} p^2 + \frac{m\omega^2}{2} q^2$$

$$= -\frac{1}{2m} \frac{m\hbar\omega}{2} (a^+ - a)^2 + \frac{m\omega^2}{2} \frac{\hbar}{2m\omega} (a^+ + a)^2$$

$$= \frac{\hbar\omega}{4} \Big\{ (a^+)^2 + a^+ a + a a^+ + a^2 - (a^+)^2$$

$$+ a^+ a + a a^+ - a^2 \Big\}$$

$$= \frac{\hbar\omega}{2} \{ a^+ a + \underbrace{a a^+}_{a a^+ = a^+ a + 1} \} = \underline{\underline{\hbar\omega \left(a^+ a + \tfrac{1}{2} \right)}}$$

c) We need

$$(a^+ a)(a^+)^n = (a^+ a) a^+ (a^+)^{n-1}$$

$$= a^+ \underbrace{(a a^+)}_{[a, a^+] = 1} (a^+)^{n-1} = a^+ (a^+ a + 1)(a^+)^{n-1}$$

$$= a^+ (a^+ a)(a^+)^{n-1} + (a^+)^n$$

$$= a^+ [(a^+ a) a^+] (a^+)^{n-2} + (a^+)^n$$

$$= (a^+)^2 (a^+ a)(a^+)^{n-2} + 2 (a^+)^n = \underset{\text{Repeat...}}{\cdots}$$

$$= n (a^+)^n$$

Hence,

$$\underline{H|n\rangle} = \frac{\hbar\omega}{\sqrt{n!}} e^{-i(n+\frac{1}{2})\omega t} (a^+a+\tfrac{1}{2})(a^+)^n|0\rangle$$

$$= \frac{\hbar\omega}{\sqrt{n!}} e^{-i(n+\frac{1}{2})\omega t} (n+\tfrac{1}{2})(a^+)^n|0\rangle$$

for t=0)

$$= \underline{\hbar\omega(n+\tfrac{1}{2})|n\rangle}$$

$$\underline{i\hbar\frac{d}{dt}|n\rangle} = \frac{i\hbar}{\sqrt{n!}}\left(\frac{d}{dt}e^{-i(n+\frac{1}{2})\omega t}\right)(a^+)^n|0\rangle$$

$$= \underline{\hbar\omega(n+\tfrac{1}{2})|n\rangle}$$

$\Rightarrow$ <u>Left hand side = Right hand side.</u>

d) We need $a(a^+)^n = (aa^+)(a^+)^{n-1}$

$$= (a^+a+1)(a^+)^{n-1}$$

$$= a^+(aa^+)^{n-2} + (a^+)^{n-1}$$

$$= \dots$$

$$= n(a^+)^{n-1}$$

Hence, $\underline{\underline{a|n\rangle}} = \frac{e^{-i(n+\frac{1}{2})\omega t}}{\sqrt{n!}} a(a^\dagger)^n |0(t=0)\rangle$

$$= \frac{e^{-i(n+\frac{1}{2})\omega t}}{\sqrt{n!}} n(a^\dagger)^{n-1} |0(t=0)\rangle$$

$$= e^{-i\omega t}\sqrt{n}\, \frac{e^{-i(n-1+\frac{1}{2})\omega t}}{\sqrt{(n-1)!}} (a^\dagger)^{n-1}|0\rangle$$

$$= \underline{\underline{e^{-i\omega t}\sqrt{n}\,|n-1\rangle}}$$

e) $a|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} a|n\rangle$

$$= e^{-|\alpha|^2/2} \sum_{n=1}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \sqrt{n}\, e^{-i\omega t} |n-1\rangle$$

$$= e^{-i\omega t} \alpha\, e^{-|\alpha|^2/2} \sum_{n=1}^{\infty} \frac{\alpha^{n-1}}{\sqrt{(n-1)!}} |n-1\rangle$$

$$= e^{-i\omega t} \alpha\, e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$= e^{-i\omega t} \alpha |\alpha\rangle$$

$\Rightarrow \quad \langle\alpha|a|\alpha\rangle = e^{-i\omega t}\alpha\langle\alpha|\alpha\rangle$

$$\underline{\langle\alpha|\alpha\rangle} = e^{-|\alpha|^2} \sum_{m=0}^{\infty}{}' \sum_{n=0}^{\infty}{}' \frac{\alpha^n \alpha^{*m}}{\sqrt{n!}\sqrt{m!}} \underbrace{\langle m|n\rangle}_{=\delta_{mn}}$$

$$= e^{-|\alpha|^2} \sum_{n=0}^{\infty}{}' \frac{|\alpha|^{2n}}{n!} = e^{-|\alpha|^2} e^{|\alpha|^2} \underline{= 1.}$$

$$\Rightarrow \quad \underline{\underline{\langle\alpha|a|\alpha\rangle = e^{-i\omega t}\alpha}}$$

$$\underline{\underline{\langle\alpha|a^\dagger|\alpha\rangle}} = \langle\alpha|a|\alpha\rangle^*$$

$$= (e^{-i\omega t}\alpha)^* \underline{\underline{= e^{i\omega t}\alpha^*}}$$

f)

$$\underline{\underline{\langle\alpha|q|\alpha\rangle}} = \langle\alpha| \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a)|\alpha\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left(e^{i\omega t}\alpha^* + e^{-i\omega t}\alpha\right)$$

$$= \sqrt{\frac{2\hbar}{m\omega}}\,\frac{1}{2}\left(e^{i\omega t - i\theta} + e^{-i\omega t + i\theta}\right)|\alpha|$$

$$= \sqrt{\frac{2\hbar}{m\omega}}\,|\alpha| \cos(\omega t - \theta)$$

$$\underline{\underline{= q_0 \cos(\omega t - \theta)}}$$

where $\underline{q_0 = \sqrt{\frac{2\hbar}{m\omega}}\,|\alpha|}$

$$\underline{\underline{\langle\alpha|p|\alpha\rangle}} = \langle\alpha|\, i\sqrt{\frac{m\hbar\omega}{2}}\,(a^+-a)\,|\alpha\rangle$$

$$= i\sqrt{\frac{m\hbar\omega}{2}}\,\langle\alpha|(a^+-a)|\alpha\rangle$$

$$= -\sqrt{\frac{m\hbar\omega}{2}}\,\frac{1}{i}\left(e^{i\omega t}\alpha^* - e^{-i\omega t}\alpha\right)$$

$$= -\sqrt{2m\hbar\omega}\,\frac{1}{2i}\left(e^{i\omega t-i\theta} - e^{-i\omega t+i\theta}\right)|\alpha|$$

$$= -\sqrt{2m\hbar\omega}\,|\alpha|\,\sin(\omega t-\theta)$$

$$= \underline{\underline{p_0 \sin(\omega t-\theta)}}$$

where

$$\underline{\underline{p_0 = -\sqrt{2m\hbar\omega}\,|\alpha|}}$$

g) We need to show that postulating a state with the property

$$a^+|\beta\rangle = \beta|\beta\rangle$$

leads to a contradiction:

1. $\langle\beta|a a^+|\beta\rangle = \underline{|\beta|^2\langle\beta|\beta\rangle}$

2. $\langle\beta|a a^+|\beta\rangle = \langle\beta|a^+ a|\beta\rangle + \langle\beta|\beta\rangle$

$$= \langle\beta| a a^{+} |\beta\rangle + \langle\beta|\beta\rangle$$

$$= |\beta|^2 \langle\beta|\beta\rangle^{*} + \langle\beta|\beta\rangle$$

$$= |\beta|^2 \langle\beta|\beta\rangle + \langle\beta|\beta\rangle = (|\beta|^2+1)\langle\beta|\beta\rangle$$

Hence we have that

$$\underline{|\beta|^2 \langle\beta|\beta\rangle = (|\beta|^2+1)\langle\beta|\beta\rangle}$$

when setting $1. = 2.$

This is <u>contradictory</u> unless <u>$|\beta\rangle = 0$.</u>

b) a is not <u>hermitean</u>.

An operator O which is hermitean fulfills the relation

$$O^{+} = O$$

Hence, if $|o_1\rangle$ and $|o_2\rangle$ are eigenstates for O:

$$O|o_1\rangle = o_1|o_1\rangle \text{ and}$$
$$O|o_2\rangle = o_2|o_2\rangle$$

Hence,

$$\langle o_1 | O | o_1 \rangle = o_1 \langle o_1 | o_1 \rangle$$

$$\langle o_1 | O | o_1 \rangle = \langle o_1 | O^\dagger | o_1 \rangle^*$$

$$= \langle o_1 | O | o_1 \rangle^* = o_1^* \langle o_1 | o_1 \rangle$$

$\Rightarrow$ $\underline{o_1 = o_1^*}$ The eigenvalues are <u>real</u>.

Suppose $\underline{o_1 \neq o_2}$

$$\langle o_1 | O | o_2 \rangle = o_2 \langle o_1 | o_2 \rangle$$

$$\langle o_1 | O | o_2 \rangle = \langle o_2 | O^\dagger | o_1 \rangle^*$$

$$= \langle o_2 | O | o_1 \rangle^* = o_1^* \langle o_2 | o_1 \rangle^*$$

$$= o_1 \langle o_1 | o_2 \rangle .$$

$\Rightarrow$

$$\underline{o_1 \langle o_1 | o_2 \rangle = o_2 \langle o_1 | o_2 \rangle}$$

$$\Rightarrow \underline{\underline{\langle o_1 | o_2 \rangle = 0}}$$

Problem 2

a) The density operator is defined as

$$\rho = \sum_{\alpha=1}^{N'} |\alpha\rangle W_\alpha \langle\alpha|$$

where $|\alpha\rangle$ is the pure state of the sub system. In our case, there are two sub-systems: $j = 1$ and 2. Both are in the ground state: $|1,1\rangle$ and $|1,2\rangle$.

We do not know wether we have a box of size 1 or 2: $W_1 = W_2 = 1/2$; minimum information.

Hence, we have

$$\rho = \frac{1}{2}\left(|1,1\rangle\langle 1,1| + |1,2\rangle\langle 1,2| \right)$$

b)

$$\underline{\underline{\rho(x,x')}} = \langle x|\rho|x'\rangle$$

$$= \frac{1}{2}\left\{ \langle x|1,1\rangle\langle 1,1|x'\rangle + \langle x|1,2\rangle\langle 1,2|x'\rangle \right\}$$

$$\underline{\underline{= \frac{1}{2}\left\{ \psi_{1,1}(x)\,\psi^*_{1,1}(x') + \psi_{1,2}(x)\,\psi^*_{1,2}(x') \right\}}}$$

c)

$$\underline{\underline{\langle H \rangle}} = \mathrm{Trace}(\rho H)$$

$$= \int_{-\infty}^{+\infty} dx \, \langle x | \rho H | x \rangle$$

$$= \int_{-\infty}^{+\infty} dx \, \langle x | \frac{1}{2} \left(|1,1\rangle\langle 1,1| H + |1,2\rangle\langle 1,2| H \right) | x \rangle$$

$$= \int_{-\infty}^{+\infty} dx \, \langle x | \frac{1}{2} \left(|1,1\rangle\langle 1,1| E_1 + |1,2\rangle\langle 1,2| E_2 \right) | x \rangle$$

$$= \int_{-\infty}^{+\infty} dx \, \frac{1}{2} \Big(E_1 \psi_{1,1}^*(x) \psi_{1,1}(x)$$

$$+ E_2 \psi_{1,2}^*(x) \psi_{1,2}(x) \Big)$$

$$= \frac{1}{2}(E_1 + E_2) = \frac{1}{2} \frac{\pi^2 \hbar^2}{2m} \left(\frac{1}{L_1^2} + \frac{1}{L_2^2} \right)$$

$$= \underline{\underline{\frac{\pi^2 \hbar^2}{4m} \left(\frac{1}{L_1^2} + \frac{1}{L_2^2} \right)}}$$