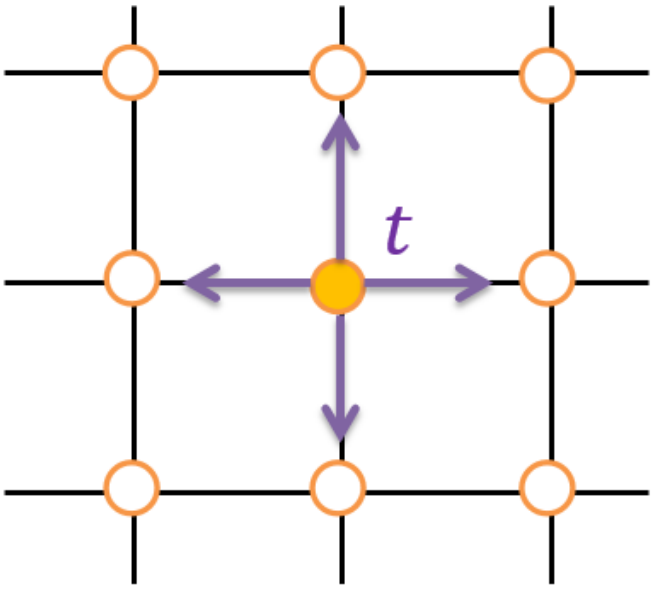
## 0.1 Additional aspects of Landau levels

### 0.1.1 Landau-levels behave differently in various materials

Landau-levels have different properties depending on which material you consider. We will show this below.

A specific material is characterized by a certain crystal structure for the atoms. This constitutes the "environment" in which the electrons move, since the electrons feel the electrostatic potential generated by the atoms in the crystal lattice. To model this, one can envision electrons hopping between different points on a lattice as shown in the figure.




Mathematically, we can model this using creation and annihilation operators: $c^\dagger$ and $c$. You will learn the formalities regarding these in courses like Many-Body Theory for Quantum Systems, but here we will just sketch how they can be used for our purpose. A creation operator $c^\dagger_{i\sigma}$ creates an electron at site $i$ with spin $\sigma$. An annihilation operator $c_{i\sigma}$ destroys an electron at site $i$ with spin $\sigma$. Thus, moving an electron from site $j$ to site $i$ can be expressed by the combination $c^\dagger_{i\sigma}c_{j\sigma}$. In other words, we describe in this way *hopping* from site $j$ to $i$. If we now associate an energy $-t$ with such a hopping process (we explain the sign of $t$ later), we can write the total Hamiltonian for the system as

$$\hat{H} = -t \sum_{\langle ij \rangle \sigma} c^\dagger_{i\sigma} c_{j\sigma}. \tag{1}$$

The brackets indicate that we should only consider hopping between nearest neighbor lattice sites, since the probability for hopping between atoms further apart should be small.

The operators $\{c^\dagger, c\}$ describe electrons which are fermions. For this reason, these operators obey different commutation properties than the bosonic $\{a, a^\dagger\}$ operators we considered in the harmonic oscillator case. For instance, the $c$-operators have to respect the Pauli principle. Therefore, they obey for instance $\{c_\lambda^\dagger, c_{\lambda'}^\dagger\} = 0$, where $\lambda$ and $\lambda'$ represent quantum numbers [for instance $\lambda = (i, \sigma)$], which expresses both the Pauli principle (for $\lambda = \lambda'$) and the antisymmetry associated with a state where two fermions have been exchanged with each other.

Let us now consider a repeating lattice structure by employing periodic boundary conditions for the $\{c^\dagger, c\}$ operators: $c_i = c_{i+N}$ (we drop the spin subscript here) where $N$ is the length of the system. This periodicity ensures that we can express the fermion operators as a Fourier-series:

$$c_i = \frac{1}{\sqrt{N}} \sum_{\boldsymbol{k}} c_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_i}. \tag{2}$$

Considering for simplicity a 1D system, we see that the only allowed $k$-values in the system are $k = 2\pi n/Na$ where $n$ is an integer. This fulfills the periodic condition $c_i = c_{i+N}$ since now $\mathrm{e}^{\mathrm{i}kNa} = \mathrm{e}^{\mathrm{i}2\pi n} = 1$. Inserting the Fourier-transform of $c_i$ into $\hat{H}$, we obtain

$$\hat{H} = \sum_k \epsilon_k c_k^\dagger c_k. \tag{3}$$

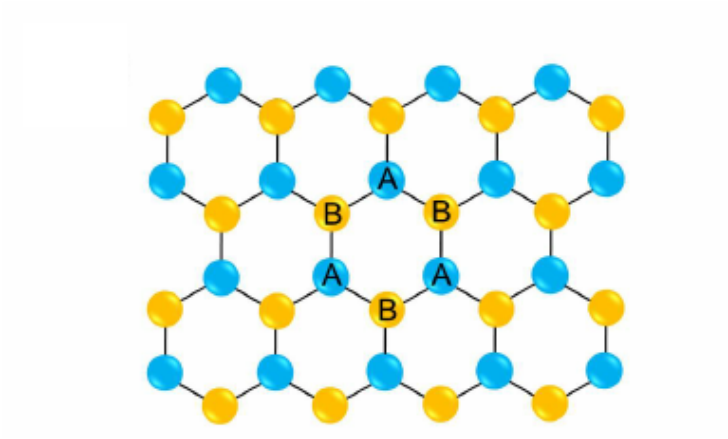
The quantity $\epsilon_k$ is the dispersion relation for the electrons and takes the form

$$\epsilon_k = -2t\cos(ka). \tag{4}$$

This is reasonable physically: our model in real space with electrons hopping around on a lattice can be expressed as a sum of electrons with kinetic energy $\epsilon_k$.

In the low-energy limit $ka \ll 1$, we get $\epsilon_k \simeq \hbar^2 k^2/2m^* - \text{const}$ where $m^* \propto 1/t$ is the effective mass. We can thus approximate our model as a free electron gas where the electrons have an effective mass determined by the hopping parameter $t$. The extra constant in the dispersion relation is not important since it simply redefines our reference level for zero energy.

The point with the above derivation was to show how the dispersion relation we use for fermions, $k^2/2m$, can arise in a material when considering its crystal structure. We have then seen how to obtain the Landau-levels when using this particular dispersion relation. But what happens when the dispersion relation in a material is fundamentally different than $k^2/2m$? Consider for instance graphene, which is a single layer of carbon atoms arranged in a honeycomb pattern.




Starting with the same hopping Hamiltonian and using periodic boundary conditions precisely as shown above, one arrives at a low-energy Hamiltonian

$$\hat{H} = \hbar v_F \boldsymbol{k} \cdot \boldsymbol{\sigma} = \hbar v_F (k_x \sigma_x + k_y \sigma_y). \tag{5}$$

Here, $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is a vector of Pauli-matrices and the Hamiltonian is thus a $2 \times 2$ matrix. This means that the wavefunction describing electrons in graphene has to be a two-component vector $\psi = (\psi_A, \psi_B)^{\mathrm{T}}$. The two components physically represent the two sublattices A and B that make up the graphene lattice.

This dispersion is very different from $k^2/2m$ and this will affect the Landau levels. With a magnetic field in the $z$-direction and using the Landau-gauge $\boldsymbol{A} = (0, Bx, 0)$, we get ($p = \hbar k$):

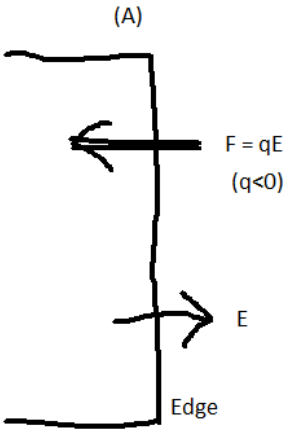
$$\hat{H} = v_F [p_x \sigma_x + (p_y - eBx)\sigma_y]. \tag{6}$$

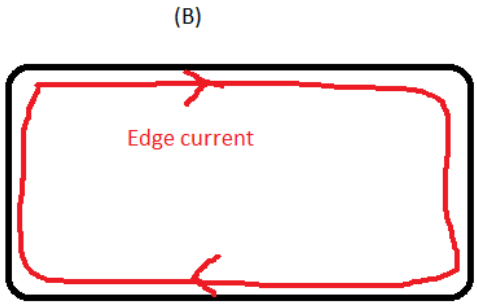
To find the Landau levels and their energies, we have to solve the Schrodinger equation $\hat{H}\psi = E\psi$. However, there is a simpler way to find the eigenvalues $E$: try to solve $H^2\psi = E^2\psi$! Which result do you get for $E$?

### 0.1.2 Edge-effects on Landau levels

When computing the number of states that fit into a given Landau level (LL), we neglected the effect that edges can have on the LL. There is a confining potential at the edges of a material which ensures that electrons do not simply flow into vacuum. Let us see how this affects the LLs.

At an edge (see figure A), there should exist an electric field $\boldsymbol{E}$ in order to have a "wall-force" acting on the electrons. The electric field exists due to the potential gradient $\nabla V$ at the edge.




Let us choose $\boldsymbol{E} = E\hat{x}$ and a gauge $\boldsymbol{A} = xB\hat{y}$ (which gives $\boldsymbol{B} = B\hat{z}$). The Hamiltonian then becomes

$$\hat{H} = \frac{1}{2m}[\hat{p}_x^2 + (\hat{p}_y + |e|Bx)^2] + |e|Ex. \tag{7}$$

The last term is the potential energy of a charge $q = -|e|$ in an electric field $E$. Since $\hat{H}$ commutes with $\hat{p}_y$, the wavefunction has to have the form $\psi = \mathrm{e}^{\mathrm{i}k_y y}\phi(x)$. We can complete the square just like we did in the case with zero electric field $E$, which gives the energies

$$E_n(k_y) = \hbar\omega_c(n + 1/2) - eE\left(k_y l_B^2 + \frac{eE}{m\omega_c^2}\right) + \frac{m}{2}\frac{E^2}{B^2}. \tag{8}$$

We defined the auxiliary quantities $\omega_c = eB/m$ and $l_B = \sqrt{\hbar/eB}$. Unlike before, the energy eigenvalues $E_n$ now depend on $k_y$: the massive degeneracy of the LLs has been lifted. As a result, the LL will drift in the $\hat{y}$-direction since the group velocity of a state is obtained from $\boldsymbol{v}_g = (1/\hbar)\partial E/\partial \boldsymbol{k}$, which gives $v_y = -E/B$.

We note that the cyclotron orbits of the LLs will drift in the direction $\boldsymbol{B} \times \boldsymbol{E}$ in general and *not* in the direction of $\boldsymbol{E}$. This can be seen even classically. In the lab frame, the force acting on the electrons is $\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) = m d\boldsymbol{v}/dt$. If we shift to a different inertial frame by letting $\boldsymbol{v} \to (\boldsymbol{E} \times \boldsymbol{B})/B^2 + \boldsymbol{v}'$, we instead obtain $m d\boldsymbol{v}'/dt = q\boldsymbol{v}' \times \boldsymbol{B}$. This describes circular motion in the new frame which is moving with $(\boldsymbol{E} \times \boldsymbol{B})/B^2$ relative the lab frame. In effect, the circular orbits are drifting.

As a consequence, we should expect an *edge current* (see figure B) in a 2D system.

*Disclaimer: first two figures taken from the web.*