TFY4205 Quantum Mechanics II
Problemset mandatory exercise 2 fall 2022

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Problem 1 (method of partial waves and Born approximation)
When particles with mass m are scattered by the repulsive potential

$$V(r) = \frac{\hbar^2 g}{2m} \frac{1}{r^2}, \tag{1}$$

where g is a dimensionless positive constant, then the scattering phase shifts are given by

$$\delta_l = \frac{\pi}{2}\left(l + \frac{1}{2} - \sqrt{(l+\frac{1}{2})^2 + g}\right). \tag{2}$$

1. What is the energy dependence of the differential scattering cross-section?
2. Find an expression for δ_l when g is small ($g \ll 1$). Use this to calculate the differential scattering cross section for small g by summing over *all* l. If you need relations for Legendre polynomials, you can certainly look them up.
3. Calculate the scattering cross-section for this potential in the first Born approximation and compare the result with point 2. (Hint: $\int_0^\infty \frac{\sin x}{x} dx = \pi/2$).
4. Derive the expression for the scattering phase shifts δ_l given above. Note that you can merge the interaction into the centrifugal potential by introducing an effective quantum number $\tilde{l}$ (instead of l) and that the solution of the radial equation without potential then has the form:

$$R_l(r) \propto \frac{\sin(kr - \tilde{l}\pi/2)}{r} \tag{3}$$

   for large r.

Problem 2
Consider scattering by a potential $V(r) = \alpha\delta(r-a)$ where α, a are both positive, real constants. The incident particle has very low energy so that we may set $ka \ll 1$ where k is the wavevector of the incident particle.

For such low energies, only the zeroth partial wave $l = 0$ should contribute to scattering. Define $a_0 = (\mathrm{e}^{\mathrm{i}\delta_0}/k)\sin\delta_0$ where δ_0 is the $l = 0$ scattering phase.

It can then be shown that the solution for the wavefunction in the region $r \geq a$ may be written as:

$$\psi = \frac{A}{kr}(\sin kr + ka_0 \mathrm{e}^{\mathrm{i}kr}) \tag{4}$$

where A is a coefficient to be determined by matching the above wavefunction to the wavefunction in the region $r \leq a$. In this inner region, the general solution of the wavefunction is

$$\psi = B\frac{\sin kr}{r} + C\frac{\cos kr}{r}. \tag{5}$$

Here, B, C are two additional coefficients to determined.

a) The quantity $a_0 = a_0(\delta_0)$ determines the differential scattering cross section in the low-energy limit. Derive an explicit expression for a_0 for the system described. Note that the expression may be simplified by using $ka \ll 1$.

b) Compute the total scattering cross section in the low-energy limit. Comment on the result you get when $\alpha \to \infty$. In particular, which physical scenario is it equivalent to?