TFY4205 Quantum Mechanics II

Problemset mandatory exercise 1 fall 2022

NTNU

Institutt for fysikk

Problem 1 (Berry's Phase for Harmonic Oscillator)

In this problem, we are going to calculate the geometric phase γ for a simple harmonic oscillator with the Hamiltonian

$$H = \frac{\hbar\omega}{2}\left[\left(P - \sqrt{2}x_2\right)^2 + \left(Q - \sqrt{2}x_1\right)^2\right] \tag{1}$$

where Q and P are the operators for position and momentum satisfying $\left[Q, P\right] = i$, and x_1 and x_2 are slowly varying real parameters that, over the course of a cycle, specify a closed curve in the x_1, x_2 plane.

a. Make use of the identity

$$e^A B e^{-A} = B + \left[A, B\right] + \frac{1}{2!}\left[A, \left[A, B\right]\right] + \ldots \tag{2}$$

which is a special case of the Baker-Campbell-Hausdorff formula (see `https://en.wikipedia.org/wiki/Baker-Campbell-Hausdorff_formula` for details and proof) to show that

$$H = D(\alpha) H_0 D^\dagger(\alpha) = \hbar\omega\left[\left(a^\dagger - \alpha^*\right)\left(a - \alpha\right) + \frac{1}{2}\right] \tag{3}$$

where

$$D(\alpha) = \exp\left(\alpha a^\dagger - \alpha^* a\right) \tag{4}$$

$$a = \frac{1}{\sqrt{2}}\left(Q + iP\right) \quad a^\dagger = \frac{1}{\sqrt{2}}\left(Q - iP\right) \quad \alpha = \left(x_1 + ix_2\right) \tag{5}$$

$$H_0 = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) \tag{6}$$

The eigenvalues and eigenstates of H_0 are, of course, $\hbar\omega(n + \frac{1}{2})$ and $|n\rangle$, respectively. The corresponding eigenstates of H are the coherent states $|n, \alpha\rangle = D(\alpha)|n\rangle$.

b. The geometric phase for the coherent state $|n, \alpha\rangle$ is given by the formula

$$\gamma_n = -\text{Im} \oint \langle n, \alpha|\nabla|n, \alpha\rangle \cdot d\mathbf{R} \tag{7}$$

$$= -\text{Im} \oint \langle n|D^\dagger(\alpha)\nabla D(\alpha)|n\rangle \cdot d\mathbf{R} \tag{8}$$

Show that

$$D^\dagger \frac{\partial D}{\partial x_1} = -ix_2 + (a^\dagger - a) \tag{9}$$

$$D^\dagger \frac{\partial D}{\partial x_2} = ix_1 + i(a^\dagger + a) \tag{10}$$

Thus, show that for any n, (8) yields

$$\gamma = \oint (x_2 dx_1 - x_1 dx_2) \tag{11}$$

What is the geometric interpretation of the integral on the right-hand side of (11)?

Problem 2 (Berry's Phase)

Assume that the time dependence of the Hamiltonian is represented by a "vector of parameters" $\mathbf{R}(t)$. That is, there exists some space in which the components of vector $\mathbf{R}(t)$ specify the Hamiltonian and change as a function of time. Therefore, we have $E_n(t) = E_n(\mathbf{R}(t))$ and $|n;t\rangle = |n(\mathbf{R}(t))\rangle$, and also

$$\langle n;t|\left[\frac{\partial}{\partial t}|n;t\rangle\right] = \langle n;t|[\nabla_R|n;t\rangle]\cdot\frac{d\mathbf{R}}{dt} \tag{12}$$

where ∇_R is simply a gradient operator in the space and direction of $\mathbf{R}$. The geometrical phase then becomes

$$\gamma_n(T) = i\int_0^T \langle n;t|[\nabla_R|n;t\rangle]\cdot\frac{d\mathbf{R}}{dt}dt \tag{13}$$

$$= \int_{\mathbf{R}(0)}^{\mathbf{R}(T)} \langle n;t|[\nabla_R|n;t\rangle]\cdot d\mathbf{R} \tag{14}$$

In the case where T represent the periods for one full cycle, so that $\mathbf{R}(T) = \mathbf{R}(0)$, where the vector $\mathbf{R}$ traces a curve C, we have

$$\gamma_n(C) = i\oint \langle n;t|[\nabla_R|n;t\rangle]\cdot d\mathbf{R}] \tag{15}$$

$$= \oint_C \mathbf{A}_n(\mathbf{R})\cdot d\mathbf{R} \tag{16}$$

$$= \int [\nabla_R \times \mathbf{A}_n(\mathbf{R})]\cdot da \tag{17}$$

such that

$$\mathbf{A}_n(\mathbf{R}) = i\langle n;t|[\nabla_R|n;t\rangle] \tag{18}$$

Show that $\mathbf{A}_n(\mathbf{R})$ is a purely real quantity.

Problem 3 (Berry's Phase)

Consider a neutron in a magnetic field. The magnetic field is fixed at an angle θ with respect to the z-axis, but rotating slowly in the ϕ-direction. That is, the tip of the magnetic field traces out a circle on the surface of the sphere at "latitude" $\frac{\pi}{2} - \theta$. Explicitly, starting out from (18), calculate the Berry potential $\mathbf{A}$ for the spin-up state w.r.t the magnetic field direction, take its curl, and determine Berry's Phase γ_+.

Hint: With a magnetic field orientated at an angle θ with respect to the z-axis, the state vector of a state parallel to this direction is

$$|n;t\rangle = \cos\left(\frac{\theta}{2}\right)|+\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|-\rangle \tag{19}$$

where $|\pm\rangle$ is a state vector orientated along $\pm z$-axis