TFY4205 Quantum Mechanics II

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Problem 1 (quantized radiation field)
In the lectures, we have quantized the vector potential as follows

$$\boldsymbol{A}(\boldsymbol{r},t) = \sum_{\boldsymbol{k},\lambda} \boldsymbol{e}_{\boldsymbol{k},\lambda} \sqrt{\frac{\hbar}{2\varepsilon_0 V \omega_{\boldsymbol{k}}}} [\hat{a}_{\boldsymbol{k},\lambda} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} + \hat{a}^\dagger_{\boldsymbol{k},\lambda} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}] \tag{1}$$

where the meaning of the symbols should be known (if not, consult the lecture notes or the textbook). In the usual Hamilton operator containing $\boldsymbol{A}$, $\hat{H} = \frac{(\hat{\boldsymbol{p}}-q\boldsymbol{A})^2}{2m}$, we see that there will exist a term $\propto \boldsymbol{A}^2$. This term, in turn, contains terms with both two creation and two annihilation operators and is thus capable of describing physical processes where two photons are created or absorbed. In other words, it can provide a finite matrix element between states that differ with two photons.

Consider now scattering of a photon on an electron at rest. The initial and final states of the system may be written as

$$|i\rangle = |\psi_i\rangle|\ldots,n_{\boldsymbol{k},\lambda},\ldots,n_{\boldsymbol{k}',\lambda'},\ldots\rangle \tag{2}$$
$$|f\rangle = |\psi_f\rangle|\ldots,n_{\boldsymbol{k},\lambda}-1,\ldots,n_{\boldsymbol{k}',\lambda'}+1,\ldots\rangle \tag{3}$$
$$\tag{4}$$

The quantities $|\psi_{i,f}\rangle$ denote the electron part of the wavefunction. Physically, this means that a photon in mode $(\boldsymbol{k},\lambda)$ has been destroyed in the final state and a mode $(\boldsymbol{k}',\lambda')$ has been created. This corresponds precisely to scattering of a photon from state $(\boldsymbol{k},\lambda)$ to state $(\boldsymbol{k}',\lambda')$.

Let M be a matrix element defined as $M \equiv \langle f|\frac{e^2}{2m}\boldsymbol{A}^2|i\rangle$. Compute the most compact expression you can for $|M|^2$.