TFY4205 Quantum Mechanics II
Problemset 4 fall 2022

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Problem 1 (WKB-approximation)
Consider a particle incident from $x < 0$ toward a potential which is equal to an unknown function $V(x)$ in the region $0 < x < a$ while the potential is zero everywhere else. The particle has an energy E which is smaller than $V(x)$ at all points in $0 < x < a$.

Set up the wavefunctions in each region of space and use appropriate boundary conditions to write down a system of linear equations for the coefficients associated with the wavefunctions (such as the transmission coefficient associated with the wavefunction in the region $x > a$). You may assume that the potential is slowly varying so that you can use the WKB-approximation.

Without solving the equations, can you say anything about which behavior you expect for one of the coefficients in the region $0 < x < a$?

Problem 2 (time-dependent perturbation theory)
The transition probability from an initial state b to a final state s is given by:

$$P_{b\to s} = |a_{b\to s}(t)|^2. \tag{1}$$

The transition amplitude, to first order in the perturbation $\hat{V}$, is given by

$$a_{b\to s} = a_s = \frac{1}{\mathrm{i}\hbar}\int_{t_0}^{t} V_{sb}(\tau)\mathrm{e}^{\mathrm{i}\omega_{sb}\tau}d\tau \tag{2}$$

when $s \neq b$. To first order in the perturbation $\hat{V}$, the transition amplitude for a system to remain in its initial state b is:

$$a_{b\to b} = 1 + \frac{1}{\mathrm{i}\hbar}\int_{t_0}^{t} V_{bb}(\tau)d\tau. \tag{3}$$

However, this transition amplitude gives a transition probability:

$$P_{b\to b} = |a_{b\to b}(t)|^2 = 1 + \frac{1}{\hbar^2}\left|\int_{t_0}^{t} V_{bb}d\tau\right|^2. \tag{4}$$

which is larger than 1. Resolve this apparent paradox.

Problem 3 (time-dependent perturbation theory)
A one-dimensional harmonic oscillator is, at $t = 0$, in the first excited state $|1\rangle$. Acting on the system is a potential that is exponentially damped as a function of time. The total Hamiltonian for $t \geq 0$ is given by

$$H = \hbar\omega(a^\dagger a + \frac{1}{2}) + V_0\mathrm{e}^{-t/\tau}(a + a^\dagger) \tag{5}$$

where τ is a constant relaxation time and $V_0 \ll \hbar\omega$.

1. What is, to first order in V_0, the wave function for $t \geq 0$?

2. After a long ("infinite") time, the energy of the system is measured. What results have the highest probability, and what are these probabilities? Consider only the eigenvalues with the three highest probabilities.