TFY4205 Quantum Mechanics II
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NTNU
Institutt for fysikk

Problem 1
To see how quantum entanglement can be used to our advantage, we construct a game around the CHSH inequality, and compare the best classical strategy to the best strategy using quantum entanglement.

The game consists of two players, Alice and Bob, which are given either a black or a white card. After looking at the card, they can choose between two doors, one to the right, and one to the left. They are not allowed to communicate during the game, but they can prepare a strategy beforehand. The rules are that they win either by both getting a black card and going through different doors, or by at least one of them getting a white card and both going through the same door.

	Different doors	Same door
Both black cards	Win	Lose
At least one white card	Lose	Win

Table 1: Winning/losing conditions

1. Classically, if they both play the game optimally, what are their chances of winning?

2. Alice and Bob have both taken Quantum Mechanics II, and know about quantum entanglement. Their new strategy consists of starting by sharing a EPR-pair before they enter the game. Their new strategy will be as follows: They can both measure two observables; Alice: A_1 and A_2, Bob: B_1 and B_2. Now, if Alice gets a white card, she measures A_1, and if the card is black, she measures A_2. Whenever these measurements give $+1$, she chooses the left door, and -1 means the right door. Bob follows the same procedure (only measuring B_1 and B_2).

	Measures $+1$	Measures -1
White card: X_1	Left door	Right door
Black card: X_2	Left door	Right door

Table 2: Choices done based on card and measurement. $X \in \{A,B\}$

Express the expectation values of A_iB_j, where $i,j \in \{1,2\}$, with $P(\mathrm{W})$ (the probability of winning) and $P(\mathrm{L})$ (the probability of losing). Combine these to get an expression that can be exploited in the CHSH inequality. From this, find the new (and better) winning probability.

[Hint: Alice and Bob wants to maximize their winning chances; is it possible to choose the "measurements angles" of A_1,A_2,B_1,B_2 in a way which accomplishes this?]

Problem 2

Quantum entanglement can be used to transfer two classical bits (*four* possible outcomes: 00, 01, 10, 11) by only transferring one quantum bit (qubit). An example of a qubit is an electron spin measured along the z-axis ($|\uparrow\rangle$ or $|\downarrow\rangle$). This is what is problem is about.

First we note that this can't be done with a single qubit, as it contains the same amount of information as a classical bit: measuring a bit gives either 0 or 1 and measuring a qubit also gives one out of two possible results. We'll need two entangled qubits make the above idea work, but importantly we only need to actually *transfer* one qubit rather than two.

Alice wants to be able to send some information to Bob, and they start by sharing an entangled pair of qubits between them. This sharing is done before Alice even knows what her message is, and the idea is that the actual message can be transmitted to Bob by sending Alices qubit to him. To encode the message, Alice performs one of the following operations on her qubit: $I, \sigma_x, \sigma_y, \sigma_z$. I is the identity operation, and $\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices. She then sends her qubit to Bob, and he does the following two measurements:

$$\sigma_x \otimes \sigma_x \quad \text{and} \quad \sigma_z \otimes \sigma_z.$$

Show that Bob can figure out which of the *four* possible measurements Alice did. The entangled pair they started out with was the EPR-pair

$$|\chi^-\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle) \tag{1}$$