From D'Alemberts principle that $\sum_i (\vec{F}_i - \dot{\vec{p}}_i) \cdot \delta \vec{r}_i = 0$ where $\delta \vec{r}_i$ is any allowed displacement of coordinate vector in the system, one derives generally:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \qquad (*)$$

where q_j is generalized coordinate j, T is kinetic energy, and Q_j is the generalized force $Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$ and $\vec{F}_i$ is the i-th component of the force in the system.

Well-known case: conservative force $\vec{F}_i = -\nabla_i V(\vec{r})$. Get:

$$Q_j = \sum_i -\frac{\partial V}{\partial r_i}\frac{\partial r_i}{\partial q_j} = -\frac{\partial V}{\partial q_j} \quad \text{(chain rule)}.$$

Since $V = V(\vec{r})$, we have $\frac{\partial V}{\partial \dot{q}_j} = 0$. Insert in $(*)$:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} - \underbrace{\frac{\partial V}{\partial \dot{q}_j}}_{=0}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \quad (L = T - V)$$

Now: what about velocity-dependent forces like Lorentz force $\vec{F} = q(\vec{v} \times \vec{B} + \vec{E})$

Generally, we can write $\vec{B} = \nabla \times \vec{A}$ and $\vec{E} = -\nabla\varphi - \frac{\partial \vec{A}}{\partial t}$ (this satisfies Maxwells laws, $\nabla \cdot \vec{B} = 0$, $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$). Insert in Lorentz:

$$F_i = -\frac{\partial U}{\partial r_i} + \frac{d}{dt}\left(\frac{\partial U}{\partial v_i}\right) \qquad (v_i = \dot{r}_i) \text{ where } U = q\varphi - q\vec{A}\cdot\vec{v}.$$

Which eqs. of motion does (**) produce for such a velocity-dep. force? Compute gen. force:

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \left[-\frac{\partial U}{\partial r_i} + \frac{d}{dt}\left(\frac{\partial U}{\partial v_i}\right)\right] \cdot \frac{\partial r_i}{\partial q_j}$$

$$= \sum_i \left[-\frac{\partial U}{\partial r_i} \cdot \partial r_i / \partial q_j + \frac{d}{dt}\left(\frac{\partial U}{\partial v_i} \cdot \frac{\partial r_i}{\partial q_j}\right) - \frac{\partial U}{\partial v_i} \cdot \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right)\right]$$

Now $v_i = \frac{dr_i}{dt} = \sum_n \frac{\partial r_i}{\partial q_n} \dot{q}_n + \frac{\partial r_i}{\partial t} \Rightarrow \frac{\partial \dot{r}_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j}$.

Moreover: $\frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right) = \frac{\partial v_i}{\partial q_j}$. Thus:

$$Q_j = \sum_i \left[-\frac{\partial U}{\partial r_i} \frac{\partial r_i}{\partial q_j} - \frac{\partial U}{\partial v_i} \cdot \frac{\partial v_i}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial v_i} \frac{\partial v_i}{\partial \dot{q}_j}\right)\right]$$

Since $U = U(q_j, \dot{q}_j) \Rightarrow Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$

Insert into (**): $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$ with $L = T - U$.

Lagrange-eqs. still hold when using a generalized velocity-dep. "potential" U.

Note that $\varphi = \varphi(\vec{r}, t)$ and $\vec{A} = \vec{A}(\vec{r}, t)$ in general. To see the min. substitut., consider now Cartesian coord. for simplicity: $L = \frac{1}{2} m \dot{x}_i \dot{x}_i + q A_i \dot{x}_i - q\varphi$.

Canonical momentum is $p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i + qA_i$. Can now construct Hamiltonian via Legendre transformation: $H = p_i \dot{x}_i - L = \frac{1}{2} m \dot{x}_i \dot{x}_i + q\varphi$.

Makes sense: kin + pot. energy. But can also write $\dot{x}_i = \frac{1}{m}(p_i - qA_i)$

$$\Rightarrow \boxed{H = \frac{1}{2m}(\vec{p} - q\vec{A})^2 + q\varphi}$$

$\vec{p}$: canonical momentum

$m\dot{\vec{x}}$: mechanical momentum.