value in the GS $|0\rangle$ : The above is an operator relation.

## Generalization

We can now generalize this to any number of fields :

$$\boxed{T\{\varphi_1 \varphi_2 \dots \varphi_n\} = :\varphi_1 \varphi_2 \dots \varphi_n: + :\text{all possible contractions}:}$$

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* Similar procedure for complex scalar field : $T\{\varphi(x)\varphi^\dagger(y)\} = :\varphi(x)\varphi^\dagger(y): + \Delta_F(x-y)$

with $\overbracket{\varphi(x)\,\varphi^\dagger(y)} \equiv \Delta_F(x-y)$ and $\overbracket{\varphi(x)\,\varphi(y)} = \overbracket{\varphi^\dagger(x)\,\varphi^\dagger(y)} = 0$ .

For instance, for $n=4$ we'd get:

$$T\{\varphi_1\varphi_2\varphi_3\varphi_4\} = :\varphi_1\varphi_2\varphi_3\varphi_4: + \overbracket{\varphi_1\varphi_2} :\varphi_3\varphi_4: + \overbracket{\varphi_1\varphi_3} :\varphi_2\varphi_4:$$

$$+ \text{ 4 similar terms}$$

$$+ \overbracket{\varphi_1\varphi_2}\,\overbracket{\varphi_3\varphi_4} + \overbracket{\varphi_1\varphi_3}\,\overbracket{\varphi_2\varphi_4} + \overbracket{\varphi_1\varphi_4}\,\overbracket{\varphi_2\varphi_3}$$

The proof of the generalization of Wick's theorem goes by induction. We have seen that it's true for $n=2$. Assume it's true for $n-1$ fields $\varphi_2 \dots \varphi_n$. If we can show that it then is true also for $n$ fields $\varphi_1\varphi_2\dots\varphi_n$, we have proven it in general.

So we add $\varphi_1$ to $\varphi_2\dots\varphi_n$ and take $x_1^0 > x_k^0$ for all $k=2,\dots n$. No loss in generality since we can move boson operators around inside $T\{\dots\}$ as much as we like and still get the same result after time-ordering. This allows us to pull $\varphi_1$ out to the left of the time-ordering:

$$T\{\varphi_1\varphi_2\dots\varphi_n\} = (\varphi_1^+ + \varphi_1^-)\,T\{\varphi_2,\dots\varphi_n\}$$

$$= (\varphi_1^+ + \varphi_1^-)\left[:\varphi_2\dots\varphi_n: + :\text{contractions}:\right] \quad (*)$$

(since the theorem is true for $n-1$ fields)

Now, we need to get $\varphi_1$ inside the normal ordering and partaking in the contractions somehow.

The $\phi_1^-$ term is easy: we can just move it into both terms since $\phi_1^- : ABCD\ldots : = :\phi_1^- ABCD\ldots :$ (recall $\phi_1^- = \int \frac{d^3k}{(2\pi)^3} a_k^+ e^{ikx} \frac{1}{\sqrt{2\omega_k}}$).

For the $\phi_1^+$ term, it has to be commuted to the right past all other $\phi$'s in order to earn a place inside the normal-ordering.

Consider first the term with no contractions.

$$\phi_1^+ : \phi_2 \cdots \phi_n : \; = \; : \phi_2 \cdots \phi_n : \phi_1^+ + [\phi_1^+, : \phi_2 \cdots \phi_n :]$$

$$= :\phi_1^+ \phi_2 \cdots \phi_n: + \underbrace{: [\phi_1^+, \phi_2] \phi_3 \cdots \phi_n + \phi_2 [\phi_1^+, \phi_3] \phi_4 \cdots \phi_n + \cdots :}$$

This follows by $\underbrace{\text{induction}}_{\text{proof via}}$ on this particular term

and by using $[A, BC] = [A, B]C + B[A, C]$.

Now in each commutator $[\phi_1^+, \phi_j] = [\phi_1^+, \phi_j^-]$ since $[\phi_1^+, \phi_j^+] = 0$.

But since $x_1^0 > x_j^0$ for all $j = 2, \ldots, n$, we have:

$$[\phi_1^+, \phi_j] = \phi_1 \phi_j \qquad (\text{since } \phi_1 \phi_j = [\phi_1^+, \phi_j^-] \text{ for } x_1^0 > x_j^0)$$

So we have found that

$$\phi_1^+ : \phi_2 \cdots \phi_n : \; = \; : \phi_1^+ \phi_2 \cdots \phi_n + \phi_1 \phi_2 \cdots \phi_n + \phi_1 \phi_2 \phi_3 \cdots \phi_n + \cdots :$$

The first term on the rhs combines with $:\phi_1^- \phi_2 \cdots \phi_n:$ to give $:\phi_1 \phi_2 \cdots \phi_n:$. That's then the first term on the rhs. of Wick's theorem for $T\{\phi_1 \phi_2 \cdots \phi_n\}$. The rest of the terms on the rhs above give all possible terms involving a single contraction of $\varphi_1$ with another field.

Next, we consider $\varphi_1^+$ : all contractions not involving $\varphi_1$ : , which is the remaining term in ($\times$). Using exactly the same procedure as we just showed for the terms in $\varphi_1^+$: all contr. not involving $\varphi_1$ : that involve one contr., this will produce all possible terms including that contraction and that contraction + a contraction of $\varphi_1$ with one of the other fields.

Doing this with all remaining terms eventually gives us all possible contractions, including the $\varphi_1$ term, which concludes the proof.