The final discrete symmetry we discuss is <u>time reversal (T)</u>.

This is slightly different from C and P as the transformation is antilinear. Unlike an ordinary linear operator $\hat{O}$ acting on Hilbert space as $O\,c|x\rangle = c\,O|x\rangle$ where c is any complex number, time-reversal acts as:

$$T\,c|x\rangle = c^{*}\,T|x\rangle$$

It has to act like this as can be seen by considering a time-independent Hamiltonian H which has states evolving in time as e^{-iHt}.

A system which at $t=0$ is in state $|\alpha\rangle$ will thus at a slightly later moment $t=\delta t$ be in state $|\alpha, t\rangle = (1-\frac{iH}{\hbar}\delta t)|\alpha\rangle$.

Note that $|\alpha\rangle$ can still depend on quantities which would be affected by a time-reversal, such as $\vec{p}$. If we now instead at moment $t=0$ apply the time-reversal operator T and then let the system evolve with time under H, the system would at $t=\delta t$ be in state $(1-\frac{iH}{\hbar}\delta t)T|\alpha\rangle$. <u>If</u> the system (H) has time reversal symmetry, this state should be the same as $T|\alpha, -\delta t\rangle$ (i.e. first look at the state at an earlier moment $t=-\delta t$ and then reverse the entire state (including e.g. momentum). Mathematically, we then have:

$$(1-\frac{iH}{\hbar}\delta t)T|\alpha\rangle = T\left[1-\frac{iH}{\hbar}(-\delta t)\right]|\alpha\rangle.$$ This is only satisfied if

$$-iHT|\alpha\rangle = TiH|\alpha\rangle.$$ If H is invariant under T, we should have $HT=TH \Rightarrow [H,T]=0$. Therefore, we need $Ti=-iT$ to fulfil the equation, proving the antilinearity. Note that time-indep. and time-reversal invariant is not the same thing in general (e.g. when including magnetic field effects).

The fact that time-reversal operation involves complex conjugation can actually be seen directly from the TDSE $i\hbar \frac{\partial \psi(\vec{x},t)}{\partial t} = H\psi(\vec{x},t)$.

Under time-reversal, H is invariant if the system has TRS (time-rev. sym.) while $\vec{x} \to \vec{x}$ and $\vec{p} \to -\vec{p}$. However, $\psi(\vec{x},-t)$ is not a solution of the resulting equation. This is due to the i on the lefthand side. We can fix this by complex conjugating ψ in addition to $t \to (-t)$. In other words, if $\psi(\vec{x},t)$ satisfies TDSE, then $\psi^*(\vec{x},-t)$ also solves the same equation given that H has TRS. So the time-reversed solution $\psi^*(\vec{x},-t)$ is obtained by $t \to (-t)$ and c.c.

We can now return to the Dirac field and the action of T on it. To do this, consider the vector current $\bar{\psi}(t,\vec{x})\gamma^\mu\psi(t,\vec{x})$: the 0-component is a charge density and should have the same form under $t \to -t$. The spatial components j_V^i are charge currents which should acquire a minus sign we run time backwards. If we now are to accomplish this, let us first write the following ansatz for the transformation of $\psi(t,\vec{x})$:

$$\psi \to T\psi(\vec{x},t)T^{-1} = B\psi'(x') = B\psi'(-t,\vec{x})$$ with a matrix B to be determined,

$x' = (-t,\vec{x})$ for TRS

we obtain

$$T\left(\bar{\psi}(t,\vec{x})\gamma^\mu\psi(t,\vec{x})\right)T^{-1} = \left[T\bar{\psi}(t,\vec{x})T^{-1}\right]\left[T\gamma^\mu T^{-1}\right]\left[T\psi(t,\vec{x})T^{-1}\right]$$

$$= T\,[\psi^\dagger(t,\vec{x})\gamma^0]\,T^{-1}\,(\gamma^\mu)^*\,B\,\psi'(-t,\vec{x})$$

$$= [\psi'(-t,\vec{x})]^\dagger B^\dagger \gamma^0 (\gamma^\mu)^* B\,\psi'(-t,\vec{x}) = \bar{\psi}'(-t,\vec{x})\,\gamma^0 B^\dagger \gamma^0 (\gamma^\mu)^* B\,\psi'(-t,\vec{x}) \quad (*)$$

Note: if you're worried about ψ not being complex conjugated by $T \ldots T^{-1}$, we will see that the matrix B is enough to give us the desired transformation for $\mathcal{L}$, i.e. B takes care of an "effective" complex conjugation. In fact, some authors in the literature use a different ansatz for the transformation, namely $\tilde{T}\psi(t,\vec{x})\tilde{T}^{-1} = \tilde{B}[\psi'(-t,\vec{x})]^*$, but then $\tilde{B}$ will be found to be different from B and $\tilde{T}$ is in fact no longer an anti-linear operator.

We can now show that it is possible to find a matrix B such that $\mathcal{L}$ and j^0_V are time-reversal invariant: ~~in the sense that acting on them with the time-reversal operator only has the effect $t \to (-t)$ and preserves their structure, i.e.~~

$$T\mathcal{L}[\psi(t)]T^{-1} = \mathcal{L} = i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi \quad \text{with } \psi = \psi(t,\vec{x}),\ \bar{\psi} = \bar{\psi}(t,\vec{x}).$$

for the Lagrangian.

Going back to $(*)$ and the vector current first, we see that we have the desired transformation properties:

$$T\,(\bar{\psi}(t,\vec{x})\,\gamma^0\,\psi(t,\vec{x}))\,T^{-1} = \bar{\psi}(t,\vec{x})\,\gamma^0\,\psi(t,\vec{x}),$$

$$T\,(\bar{\psi}(t,\vec{x})\,\gamma^i\,\psi(t,\vec{x}))\,T^{-1} = -\,\bar{\psi}(t,\vec{x})\,\gamma^i\,\psi(t,\vec{x})$$

if we choose B so that the relations

$$\gamma^0 B^\dagger \gamma^0 (\gamma^0)^* B = \gamma^0 \quad \text{and} \quad \gamma^0 B^\dagger \gamma^0 (\gamma^i)^* B = -\gamma^i$$

both hold along with $\psi'(-t, \vec{x}) = \psi(t, \vec{x})$.

Since $(\gamma^0)^* = \gamma^0$, the first one tells us that B is unitary: $B^\dagger = B^{-1}$.

Together with the second one, the correct solution is deduced to be:

$B = e^{i\varphi}\gamma^1\gamma^3$ where $e^{i\varphi}$ is an arbitrary phase. Perhaps doing

two TRS will give us a condition on what $e^{i\varphi}$ has to be,

since two TRS should be the same as doing nothing.

Surprisingly, this is not the case:

$$T\left[T(\psi(t,\vec{x}))T^{-1}\right]T^{-1} = T\left[e^{i\varphi}\gamma^1\gamma^3\psi'(-t,\vec{x})\right]T^{-1}$$

$$= e^{-i\varphi}\gamma^1\gamma^3\, T(\psi'(-t,\vec{x}))T^{-1} = e^{-i\varphi}\gamma^1\gamma^3 e^{i\varphi}\gamma^1\gamma^3\psi(t,\vec{x}) = \underline{-\psi(t,\vec{x})}$$

Not only does the phase remain arbitrary (so let's set it to unity),

but ψ has acquired an overall $-$ sign we can't get rid off!

This is in fact a general property for T acting on fermions:

$$T^2 = \begin{cases} +1 & \text{for bosons (integer spin)} \\ -1 & \text{for fermions (half-integer spin)} \end{cases}$$

Fortunately, $\psi(x)$ is in itself not an observable and all physical operators come with an even number of Dirac fields, so the extra sign won't show up there.

Let's now confirm that $\mathcal{L}$ is invariant under time-reversal!

The mass term:

$$T(m\bar\psi\psi)T^{-1} = m\,T\psi^\dagger T^{-1}\gamma^0 T\psi T^{-1} = m[\psi'(-t,\vec x)]^\dagger \gamma^{3\dagger}\gamma^{1\dagger}\gamma^0\gamma^1\gamma^3\psi'(-t,\vec x)$$

$$= m\bar\psi(t,\vec x)\psi(t,\vec x) \qquad | \text{ by using } \{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}$$

satisfies this and since

$$T(i\partial^0)T^{-1} = (-i)(-\partial_0) = i\partial_0 \quad \text{and} \quad T(i\partial_i)T^{-1} = -i\partial_i \quad (*)$$

the kinetic term also satisfies

$$T(i\bar\psi\partial\!\!\!/\psi)T^{-1} = i\bar\psi(t,\vec x)\,\partial\!\!\!/\,\psi(t,x).$$

This is seen by combining $(*)$ with the way the vector current j^μ_V transformed.

Inserting a γ^5 to get the axial current j^μ_A does not spoil TRS since γ^5 is real and commutes with $\gamma^1\gamma^3$. Therefore, chiral Lagrangians $i\bar\psi_{L,R}\partial\!\!\!/\psi_{L,R}$ are also T-invariant (as they are obtained with projection operators involving γ^5).

Everything seems to be T-invariant so far. However, there does exist a Hermitian bilinear (and thus in principle observable) combination that is not T-invariant:

$$i\bar\psi\gamma^5\psi.$$

This term is called a pseudoscalar because it is invariant under continuous (proper) Lorentz transformations, but changes sign under parity. The factor i is necessary for this term to be Hermitian. Under $T\mathcal{R}$, we have:

$$T(i\bar{\psi}\gamma^5\psi)T^{-1} = -i\,\bar{\psi}(t,\vec{x})\,\gamma^5\,\psi(t,\vec{x})$$

by using $[\gamma^5, B]=0$. Actually, this combination $(i\bar{\psi}\gamma^5\psi)$ also changes sign under CP (see exercise):

$$CP\,(i\bar{\psi}\gamma^5\psi)P^{-1}C^{-1} = -i\,\bar{\psi}(t,\vec{x})\,\gamma^5\,\psi(t,x)$$

For a long time, it was thought that there was no CP violation in nature. But in the early 1960's, CP violation was discovered in rare kaon decays.

The combination CPT is nevertheless believed to be a good symmetry of nature, as it is not possible to write down a Lorentz-invariant, Hermitian Lagrangian that violates CPT.

Since CPT should hold, CP violation implies T violation, and some experiments claim to have observed this.