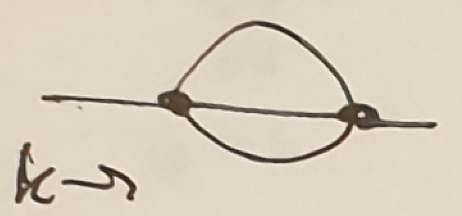
Using $G_1 = \dfrac{}{l_{1E}^2+m^2}$, $G_2 = \dfrac{}{l_{2E}^2+m^2}$, $G_3 = \dfrac{}{(k_E-l_{1E}-l_{2E})^2+m^2}$, we may then

write :



$$= \frac{i\lambda^2 \mu^{8-2D}}{6} \iint \frac{d^D l_{1E}\, d^D l_{2E}}{(2\pi)^D (2\pi)^D} \int_0^\infty d\rho \cdot \rho^2 \cdot \int_0^1\!\!\int_0^1\!\!\int_0^1 dx_1 dx_2 dx_3$$

$$\times \delta(1-x_1-x_2-x_3) \cdot \exp\left[-\rho\left(x_1 l_{1E}^2 + x_2 l_{2E}^2 + x_3 (k_E - l_{1E} - l_{2E})^2 + m^2\right)\right]. \qquad (*)$$

We used that $(x_1+x_2+x_3)m = m$ in the integrand due to the δ-function and that

$$\int_0^\infty \rho^2 e^{-\rho b} \cdot d\rho = \frac{2}{b^3}.$$

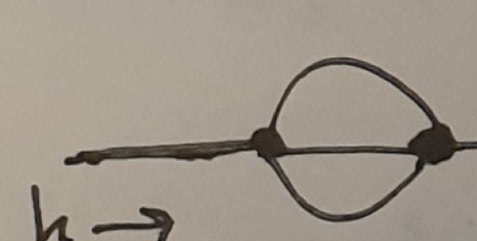
At this stage, we now justify the Wick rotation done initially by showing that we get exactly $(*)$ by first doing Feynman parametrization and then Wick rotation $\Rightarrow$

To proceed, we complete the square for l_{1E}:

$$x_1 l_{1E}^2 + x_3 (k_E - l_{1E} - l_{2E})^2 = (x_1+x_3)\left[l_{1E} + \frac{x_3}{x_1+x_3}(l_{2E}-k_E)\right]^2 + \frac{x_1 x_3}{x_1+x_3}(l_{2E}-k_E)^2$$

and then do the Gaussian integral over l_{1E} by using $\tilde{l}_{1E} \equiv l_{1E} + \frac{x_3}{x_1+x_3}(l_{2E}-k_E)$

$$\int d^D \tilde{l}_{1E} = \int d^D l_{1E} \quad \text{and} \quad \int d^D \tilde{l}_{1E}\, e^{-a \tilde{l}_{1E}^2} = \left(\frac{\pi}{a}\right)^{D/2}.$$

$\Rightarrow$ 

$$= \frac{i\lambda^2 \mu^{8-2D}}{6\cdot(4\pi)^{D/2}} \int \frac{d^D l_{2E}}{(2\pi)^D} \int_0^\infty \rho^{2-D/2} d\rho \int_0^1 \frac{dx_1 dx_2 dx_3}{(x_1+x_3)^{D/2}} \delta(1-x_1-x_2-x_3)$$

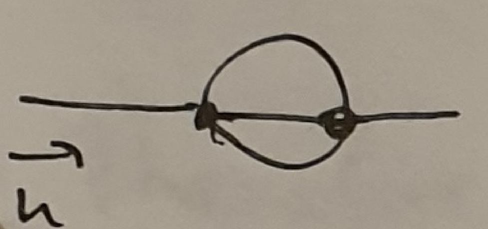
$$\times \exp\left[-\rho\left(x_2 l_2^2 + \underbrace{\left(x_3 - \frac{x_3^2}{x_1+x_3}\right)}_{= \frac{x_1 x_3}{x_1+x_3}} (k_E - l_2)^2 + m^2\right)\right]$$

We now complete the square for l_{2E}:

$$x_2 l_{2E}^2 + \left(x_3 - \frac{x_3^2}{x_1+x_3}\right)(k_E - l_{2E})^2 = \frac{x_1x_2+x_2x_3+x_3x_1}{x_1+x_3}\left[l_2 - \frac{X}{x_2}k_E\right]^2 + X k_E^2$$

with X defined as $\displaystyle X \equiv \frac{x_1x_2x_3}{x_1x_2+x_2x_3+x_3x_1}$.

Doing the integral over l_{2E} in the same way as we did for l_{1E}:



$$= \frac{i\lambda^2\mu^{8-2D}}{6\cdot(4\pi)^D}\int_0^\infty \rho^{2-D}\,d\rho \int_0^1 \frac{dx_1dx_2dx_3}{(x_1x_2+x_2x_3+x_3x_1)^{D/2}}\,\delta(1-x_1-x_2-x_3)\,e^{-\rho(Xk_E^2+m^2)}$$

and then doing the integral over ρ by utilizing that it is essentially a Γ-function:

$$= \frac{i\lambda^2\mu^{8-2D}}{6\cdot(4\pi)^D}\,\Gamma(3-D)\int_0^1 \frac{dx_1dx_2dx_3}{(x_1x_2+x_2x_3+x_3x_1)^{D/2}}\,\delta(1-x_1-x_2-x_3)\,(m^2-Xk^2)^{D-3}$$

where we in the last step replaced k_E^2 with $-k^2$. For $D=4$, this diverges (as we know from the beginning) since $\Gamma(-1)$ has a pole.

But now we use the dim. reg. and set $D=4-2\varepsilon$. Then, we have

$\displaystyle \Gamma(3-D) = \Gamma(-1+2\varepsilon) = \frac{\Gamma(2\varepsilon)}{2\varepsilon-1} \approx -\frac{1}{2\varepsilon}$. Moreover, we use the expansion

$$(m^2 - Xk^2)^{1-2\varepsilon} \approx m^{-4\varepsilon}\left[1-2\varepsilon\cdot\ln\left(1-\frac{Xk^2}{m^2}\right)\right](m^2-Xk^2) .$$

Considering the first term in the square brackets, $m^{-4\varepsilon} \cdot 1 \cdot (m^2 - X k^2)$,

we will find below that a singularity arises from the integrals over the Feynman parameters if $m^2 \neq 0$. This happens in the ~~three~~ corners of the integration region where two of the Feynman parameters x_i approach 0.

Note that in this case $X \to 0$, so the k^2 term does not contribute to the singularity.

Therefore, we only need to compute the (divergent) part of the integral, namely:

$$\int_0^1\int_0^1\int_0^1 \frac{dx_1\, dx_2\, dx_3}{(x_1x_2 + x_2x_3 + x_3x_1)^{2-\varepsilon}}\, \delta(1-x_1-x_2-x_3) \approx \int_0^a\int_0^a \frac{dx_1\, dx_2}{(x_1+x_2)^{2-\varepsilon}} + \int_0^a\int_0^a \frac{dx_2\, dx_3}{(x_2+x_3)^{2-\varepsilon}} + \int_0^a\int_0^a \frac{dx_1\, dx_3}{(x_1+x_3)^{2-\varepsilon}} \quad *$$

where we used the δ-function to get rid off one integral and then only kept linear terms in x_i since these contribute the most near the corners where two $x_i \to 0$.

We thus have:

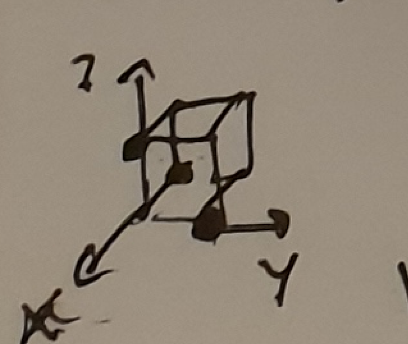
$$\int_0^1 \frac{dx_1\, dx_2\, dx_3}{(\dots)^{2-\varepsilon}} \approx 3 \int_0^a \frac{dx_1\, dx_2}{(x_1+x_2)^{2-\varepsilon}} = 3 \cdot \frac{(2^\varepsilon - 2)\, a^\varepsilon}{(\varepsilon - 1)\cdot \varepsilon} \overset{\lim_{\varepsilon \to 0}}{\approx} \frac{3}{\varepsilon}$$

(use math integrator or lookup)

This leads to a Δm^2 term with a double pole in ε:

$$\Delta m^2 = \frac{\lambda^2 m^2}{4 \cdot (4\pi)^4} \cdot \frac{1}{\varepsilon^2} + O\left(\frac{1}{\varepsilon}\right)$$

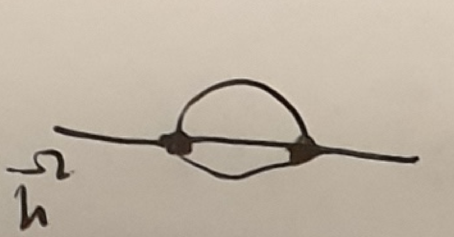
Here, $a \ll 1$. This can be understood as follows. The divergence occurs in 3 corners of the region:



The contribution from one such corner (say $x_3 = 0$) occurs for $x_3 \approx 1-a$ to $x_3 = 1$ where $a \ll 1$, while x_1 and x_2 vary from 0 to a. Thus: integral $= \int_0^a \int_0^a \int_{1-a}^1 \dots + (x_2, x_3 \to 0 \text{ corner}) + (x_1, x_3 \to 0 \text{ corner})$

where we used that $\left(\frac{\mu}{m}\right)^{2\varepsilon} \simeq 1 + \varepsilon \ln\left(\frac{\mu^2}{m^2}\right)$.

We've now computed one part of the integral: we had



$$= \frac{i\lambda^2 \mu^{8-2D}}{6\cdot(4\pi)^D}\,\Gamma(3-D)\int_0^1 \frac{dx_1dx_2dx_3}{(x_1x_2+x_2x_3+x_3x_1)^{2-\varepsilon}}\,\delta(1-x_1-x_2-x_3)\cdot m^{-4\varepsilon}\left[1-2\varepsilon\ln\left(1-\frac{Xk^2}{m^2}\right)\right](m^2-Xk^2)$$

and computed the contribution from the part $\propto m^{-4\varepsilon}\cdot 1\cdot m^2$. There are thus two remaining parts $\propto -m^{-4\varepsilon}\cdot 1\cdot Xk^2$ and $\propto -m^{-4\varepsilon}\cdot 2\varepsilon\cdot\ln\left(1-\frac{Xk^2}{m^2}\right)\cdot(m^2-Xk^2)$.

The first one gives

$$\frac{i\lambda^2\mu^{8-2D}}{6\cdot(4\pi)^D}\,\Gamma(3-D)\int_0^1 \cdots\cdot\left(-m^{-4\varepsilon}\cdot Xk^2\right) = -\frac{i\lambda^2\mu^{4\varepsilon}m^{-4\varepsilon}\cdot\Gamma(-1+2\varepsilon)}{12\cdot(4\pi)^{4-2\varepsilon}}\,(1+\varepsilon C)\,k^2 \quad (*)$$

where we defined the (finite) integral:

$$C \equiv 2\cdot\int_0^1 \frac{dx_1dx_2dx_3}{(x_1x_2+x_2x_3+x_3x_1)^3}\cdot\ln(x_1x_2+x_2x_3+x_3x_1)\,\delta(1-x_1-x_2-x_3).$$

PROOF of (*): We use $\frac{1}{a^{3-\varepsilon}} = \frac{a^\varepsilon}{a^3} \simeq \frac{1+\varepsilon\ln(a)}{a^3}$ for $\varepsilon \ll 1$ to get:

$$\frac{i\lambda^2\mu^{8-2D}\,\Gamma(3-D)}{6\cdot(4\pi)^D}\int_0^1 \frac{dx_1dx_2dx_3\;x_1x_2x_3}{(x_1x_2+x_2x_3+x_1x_3)^{3-\varepsilon}}\cdot\left(-m^{-4\varepsilon}\cdot k^2\right)$$

$$= -i\,\frac{\lambda^2 \mu^{4\varepsilon} m^{-4\varepsilon}\,\Gamma(-1+2\varepsilon)\cdot k^2}{6\cdot(4\pi)^{4-2\varepsilon}} \cdot \int_0^1 \frac{dx_1 dx_2 dx_3\, x_1 x_2 x_3}{(x_1 x_2 + \dots)^3} \cdot \left(1 + \varepsilon \ln\left[(x_1 x_2 + \dots)\right]\right)\cdot \delta(1 - \dots)$$

We look at the first part of the integral first:

$$\int_0^1\int_0^1\int_0^1 \frac{dx_1 dx_2 dx_3\, x_1 x_2 x_3}{(x_1x_2 + x_1x_3 + x_2x_3)^3}\,\delta(1-x_1-x_2-x_3)$$

$$= \int_0^1 \int_0^{1-x_1} \frac{dx_1 dx_2\, x_1 x_2 (1-x_1-x_2)}{(x_1+x_2-x_1x_2-x_1^2-x_2^2)^3} = \frac{1}{2} \quad \text{(math integrator)}.$$

Note how we have to replace the integration limit of the x_2-integral when using the δ-function, because otherwise we are taking into account a region in the (x_1, x_2)-plane which is not reconcilable with the point where we used the δ-function (such as $x_1 \approx 1, x_2 \approx 1$). Second part of integral is ok.

We now expand (*) for small ε as before and get:

$$\frac{i\lambda^2}{12(4\pi)^4}\left(\frac{1}{2\varepsilon} + 1 - \gamma_E + \ln(4\pi) - \frac{C}{2} + \ln\left(\frac{\mu^2}{m^2}\right)\right)k^2 + O(\varepsilon).$$

Now, there is just one more term remaining to evaluate, namely

$$\frac{i\lambda^2 \mu^{8-2D}}{6\cdot(4\pi)^D}\,\Gamma(-1+2\varepsilon)\int_0^1 \frac{dx_1 dx_2 dx_3}{(\dots)^{2-\varepsilon}}\,\delta(1-\dots)\cdot\left[-m^{-4\varepsilon}\cdot 2\varepsilon\cdot \ln\left(1-\frac{Xk^2}{m^2}\right)\left(1-\frac{Xk^2}{m^2}\right)\right]\cdot m^2$$

Since $\Gamma(-1+2\varepsilon) \approx -\frac{1}{2\varepsilon}$, we get:

$$\frac{i\lambda^2 m^2}{6\cdot(4\pi)^4}\cdot f\left(\frac{k^2}{m^2}\right)$$ where we defined:

$$f(z) = \int_0^1 \frac{dx_1\,dx_2\,dx_3}{(x_1x_2 + x_2x_3 + x_3x_1)^2}\,(1-Xz)\cdot\ln(1-Xz)\,\delta(1-x_1-x_2-x_3)$$

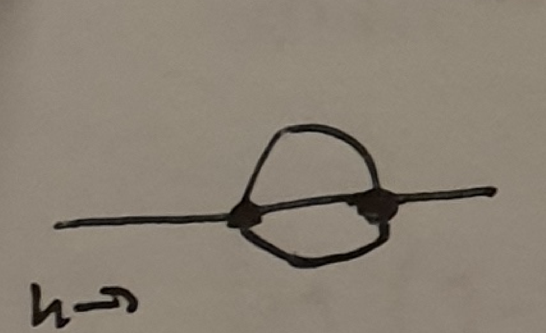
where we could let $2-\varepsilon \to 2$ in the exponent because the integral does not diverge. Can be seen by considering that:

$$f(z) = \int_0^1\int_0^{1-x_1} \frac{dx_1\,dx_2\,(1-Xz)\ln(1-Xz)}{(x_1+x_2-x_1x_2-x_1^2-x_2^2)} \quad \text{where now } X = \frac{x_1x_2(1-x_1-x_2)}{(x_1+x_2-\dots)}$$

At the problematic points $(x_1\to 0, x_2\to 1)$, $(x_1\to 1, x_2\to 0)$, $(x_1\to 0, x_2\to 0)$, we always have $X\to 0$ so that $\ln(1-Xz)\simeq -Xz$ and the ratio

$$\frac{X}{(\dots)} \text{ and } \frac{X^2}{(\dots)}$$ never diverges at these points.

We have now evaluated the complete sunset diagram integral:



$$= \frac{-i\lambda^2 m^2}{4\cdot(4\pi)^4\cdot\varepsilon^2} + \frac{i\lambda^2}{12\cdot(4\pi)^4}\left(\frac{1}{2\varepsilon} + 1 - \gamma_E + \ln(4\pi) - \frac{C}{2} + \ln\left[\frac{\mu^2}{m^2}\right]\right)k^2$$

$$+ \frac{i\lambda^2}{6\cdot(4\pi)^4}\, m^2 f\left(\frac{k^2}{m^2}\right)$$

With the final result in hand, we now go back to a statement we made before we started to solve the integral and Wick rotate : extra care had to be taken when k^2 exceeded a threshold value. Let's now see why this is the case.

The expression for $-i\Sigma$ depended on a function $f\left(\frac{k^2}{m^2}\right)$ were

$$f(z) = \int_0^1 \frac{dx_1\, dx_2\, dx_3\, \delta(1-x_1-x_2-x_3)}{(x_1 x_2 + x_2 x_3 + x_3 x_1)^2} \quad (1-Xz)\ln(1-Xz)$$

This is real for $z<9$, but gets an imaginary part for $z>9$. This is because the argument of $\ln$ is negative in part of the integration region if $z>9$. This follows since X has a maximum value of $\frac{1}{9}$, reached when $x_1 = x_2 = x_3 = \frac{1}{3}$.

Assume that we (as is natural) choose the branch-solution of $\ln$ such that it has no imaginary part if $z<9$ [recall $\ln(re^{i\theta}) = \ln r + i(\theta + 2\pi$

i.e. its principal value with imaginary part $\in (-\pi, \pi]$.

Then, if z has a small positive imag. part, the imaginary part of the $\ln$ is $-i\pi$ when the real part of the argument is less than zero. To see this, let $r>1$ so that:

$$\ln(\underbrace{1-z}_{r+i\varepsilon}) = \ln(1-\underbrace{re^{i\varepsilon}}_{=r+i\varepsilon}) = \ln[\underbrace{(1-r)e^{i\varepsilon}}_{1-r-i\varepsilon}] = \ln[(r-1)e^{i(\varepsilon+\pi)}]$$

since $1-r<0$.

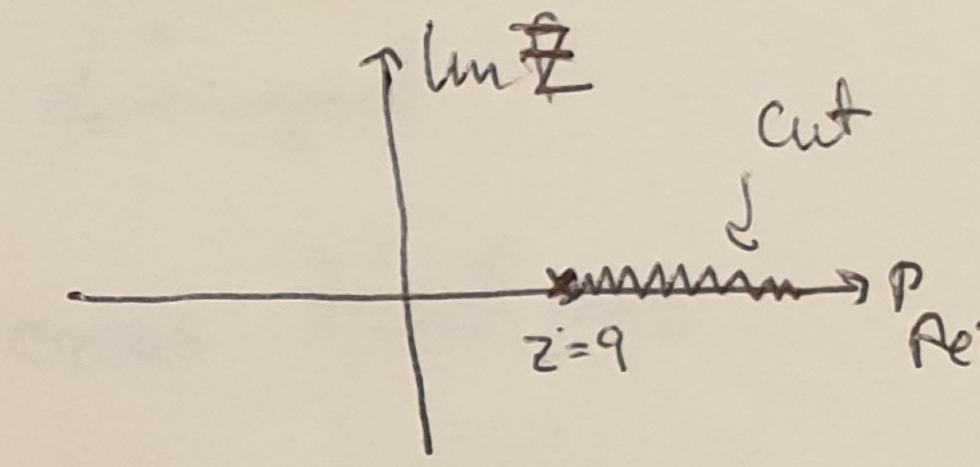
$$= \underbrace{\ln(r-1)}_{>0} + i(\varepsilon+\pi) - \underbrace{2\pi i}_{\text{to remain on the principal branch}} = \ln(r-1) - i\pi.$$

On the other hand, if z has a small imaginary negative part, the imag. part of $\ln$ is $+i\pi$. This is seen from above by $\varepsilon \to (-\varepsilon)$ and noting that we then don't have to subtract $2\pi i$ to remain in the selected branch of $\ln$.

Therefore, if z is real we may write:

$$\mathrm{Im}[f(z \pm i\varepsilon)] = \pm \pi \int_0^1 \frac{dx_1 dx_2 dx_3\, \delta(1-x_1-x_2-x_3)}{(x_1x_2+x_2x_3+x_3x_1)^2} (Xz-1)\cdot \Theta(Xz-1)$$

which is non-zero only if $z>9$ and thus is discontinuous across the real axis for $z>9$: $f(z)$ has a branch cut as follows and $z=9$ is a branch point ($x=0$ is b.p. for $\ln x$)



To figure out if we should take the value of f above or below the branch cut, the $i\varepsilon$ hidden in m^2 comes to the rescue. We reinstate it as $m^2 \to m^2 - i\varepsilon$, which causes $z \to z + i\varepsilon$ since $f\left(\frac{k^2}{m^2}\right) \to f\left(\frac{k^2}{m^2} + i\varepsilon\right)$, for $k^2 > 0$, so we should use $\mathrm{Im}\{f(\frac{k^2}{m^2})\} = +\pi \int_0^1 dx_1 \ldots$.

Let's now carefully work out which counterterms we have to add and what the remaining physical self-energy and propagator is.

We know that the counterterm $-i\Sigma_{counter} = \times = i\left[\delta_Z(k^2-m^2)-\delta_{m^2}\right]$ will cancel any divergence in the k^2-m^2 term in the propagator if we choose $\delta_Z = \dfrac{\partial}{\partial(k^2)}\Sigma_0(k^2)\Big|_{k^2=m^2}$ (*),

Since this effectively shifts the self-energy to:

$$\Sigma_0 \to \Sigma_0 - \delta_Z(k^2-m^2) + \delta m^2 .$$

Then, δm^2 should be chosen so that any remaining divergent terms vanish.

In other words: we should choose δ_Z and δ_{m^2} so that all divergent terms in the Σ_0-term are cancelled.

In the OS renorm. scheme, the finite part of counterterms is chosen so that the renormalized mass is the physical one.

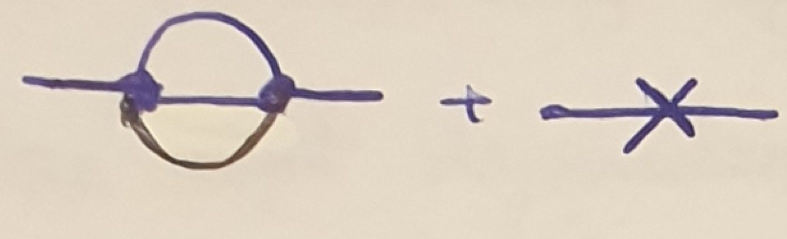
For the sake of argument, therefore let's say that we wanted to fix the pole to lie at the bare mass m^2 and have residue 1 there. We consider here only the contribution to Σ from the sunset diagram.

First, observe that

$$\delta_Z = \frac{-\lambda^2}{12(4\pi)^4}\left[\frac{1}{2\varepsilon} + 1 - \gamma_E + \ln(4\pi) - \frac{c}{2} + \ln\left(\frac{\mu^2}{m^2}\right) + 2f'(1)\right]$$

(*) and also keep the residue at the pole equal to 1 (if m is the actual physical mass). In general, this is known as the on-shell renormalization scheme condition: $\delta_Z = \dfrac{\partial\Sigma_0}{\partial(k^2)}\Big|_{k^2=m^2_{phys}}$

Therefore, the total self-energy $\underbrace{}_{-i\Sigma_{tot}}$ including counterterms is:



$$= -\frac{i\lambda^2 m^2}{4\cdot(4\pi)^4\cdot\varepsilon^2} - \frac{i\lambda^2}{6\cdot(4\pi)^4}\left[-m^2 f\left(\frac{k^2}{m^2}\right) + (k^2-m^2)f'(1) - \frac{m^2 P}{2}\right]$$

$$-i\delta m^2.$$

Here, we defined $P \equiv \left[\frac{1}{2\varepsilon} + 1 - \gamma_E + \ln(4\pi) - \frac{c}{2} + \ln\left(\frac{\mu^2}{m^2}\right)\right]$. Thus:

$$\Sigma_{tot} = \frac{\lambda^2}{6\cdot(4\pi)^4}\left[(k^2-m^2)f'(1) - m^2 f\left(\frac{k^2}{m^2}\right)\right] + \underbrace{\frac{\lambda^2 m^2}{4\cdot(4\pi)^4\cdot\varepsilon^2} - \frac{\lambda^2 m^2 P}{12\cdot(4\pi)^4}}_{\text{divergent terms}} + \delta m^2.$$

It remains to choose δm^2 so that the (i) divergent terms (and unphysical dependence on μ) are cancelled and (ii) so that $\Sigma_{tot}(k^2=m^2)=0$, ensuring that the pole is at $k^2=m^2$.

We satisfy both of these criteria by choosing

$$\delta m^2 = -\left(\text{divergent terms}\right) + \frac{\lambda^2}{6\cdot(4\pi)^4} m^2 \cdot f(1),$$ leaving us with

$$\Sigma_{tot} = \frac{\lambda^2}{6\cdot(4\pi)^4}\left[m^2 f(1) + (k^2-m^2)f'(1) - m^2 f\left(\frac{k^2}{m^2}\right)\right].$$

This satisfies $\Sigma_{tot}(k^2=m^2)=0$ and has no divergent terms <u>or</u> dependence on the unphysical μ.

We underline again that δm^2 and δ_Z should generally be chosen to remove any divergences/unphysical terms, but finite terms can be kept depending on which renormalization scheme one uses.

For instance, the finite part of ... its environment. the counterterms are chosen so that (their finite parts) the renormalized mass is the physical mass when using our on-shell renorm. scheme. Other schemes (MS) are possible, but then the renom. parameters <u>are not</u> the phys. ones.

4-point correlators and the effective coupling

We will now consider 4-point correlators up to 1-loop level, as this will show how not only masses and residues (amplitude/normalization & the momenta) are renormalized, but also the <u>coupling</u> strength