$\mathcal{O}(t) \equiv e \quad \mathcal{O} e$ while W_I goes

Apply this now to the op. $e^{iH_0T} e^{-2iHT} e^{iH_0T}$. First re-express as:

$$e^{iH_0T} e^{-2iHT} e^{iH_0T} = \lim_{\Delta t \to 0} e^{iH_0T} \left(e^{-iH_0 \Delta t} e^{-iH_I \Delta t} \right)^{\frac{2T}{\Delta t}} e^{iH_0T}$$

with $H = H_0 + H_I$. Then use: $e^{-iH_0\Delta t} = e^{-iH_0(t+\Delta t)}\, e^{iH_0 t}$ where the t is determined by the position of the particular $e^{-iH_0\Delta t}$ in the product.

We then have:

$$e^{iH_0T}e^{-2iHT}e^{iH_0T} = \lim_{\Delta t\to 0} e^{iH_0T}\left[e^{-iH_0(t_1+\Delta t)}\, e^{iH_0t_1}\, e^{-iH_I\Delta t}\right]$$

$$\times\left[e^{-iH_0(t_2+\Delta t)}\, e^{iH_0t_2}\, e^{-iH_I\Delta t}\right]\times\dots\left[e^{-iH_0(t_n+\Delta t)}\, e^{iH_0t_n}\, e^{-iH_I\Delta t}\right]e^{iH_0T}$$

Here, we choose t_j so that $t_1 = T,\ t_2 = T-\Delta t, \dots t_n = -T.$ $^{(*)}$

Since the first three factors $e^{iH_0T}\cdot e^{-iH_0(t_1+\Delta t)} \to 1$ in the limit $\Delta t \to 0$,

$$e^{iH_0T}\cdot e^{-2iHT}\cdot e^{iH_0T} = \lim_{\Delta t\to 0}\prod_t\left(e^{iH_0t}\, e^{-iH_I\Delta t}\, e^{-iH_0t}\right)$$

$$= T\left\{\lim_{\Delta t\to 0}\prod_t e^{-iH_I(t)\Delta t}\right\}$$ | since the t's are time-ordered from the outset.

$$= T\left\{e^{-i\int dt\, H_I(t)}\right\}$$

$$= T\left\{e^{i\int d^4x\, \mathcal{L}_I}\right\}$$ | since $L_I = \int d^3x\, \mathcal{L}_I = -H_I$. $^{(+)}$

Therefore, we have proven that for any states, we have:

$$\langle\psi|e^{iH_0T}e^{-2iHT}e^{iH_0T}|\varphi\rangle = \langle\psi|T\{e^{i\int d^4x\,\mathcal{L}_I}\}|\varphi\rangle$$

where the time-dependence of $\mathcal{L}_I$ is then determined by H_0 (just like we're done all the way up to now since we use the same field expansion of $\varphi(x)$ in terms of $a_k e^{-ikx}$ etc. in $\mathcal{L}_I$ despite the presence of interaction. So it's the same $\mathcal{L}_I(x)$ that we're used to).

(+) $L_I = -H_I$ is true when L_I does not depend on derivatives of fields and when L_0 is quadratic in the fields: $\mathcal{L}[\varphi, \partial\varphi] = \frac{1}{2}(\partial\varphi)^2 - \mathcal{L}_{int}[\varphi] \Rightarrow \mathcal{H} = \partial\varphi\frac{\delta\mathcal{L}}{\delta(\partial\varphi)} - \mathcal{L}[\varphi,\partial\varphi] = \frac{1}{2}(\partial\varphi)^2 + \mathcal{L}_{int}[\varphi]$.

(*) This is allowed since the choice of t's is arbitrary, so we choose them to be time-ordered.