NTNU Trondheim, Institutt for fysikk

Examination for FY3464 Quantum Field Theory I
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Allowed tools: mathematical tables

1. Noether's theorem.

Show that a continuous global symmetry of a set of fields ϕ_a described by a Lagrangian $\mathscr{L}(\phi_a, \partial_\mu\phi_a)$ leads classically to the conserved current

$$j^\mu = \frac{\delta\mathscr{L}}{\delta\partial_\mu\phi_a}\,\delta\phi_a - K^\mu\,, \tag{1}$$

where K^μ is a four-divergence, $\delta\mathscr{L} = \partial_\mu K^\mu$. (6 pts)

We assume first that $\mathscr{L}(\phi_a, \partial_\mu\phi_a)$ us invariant under an infinitesimal change $\delta\phi_a$,

$$0 = \delta\mathscr{L} = \frac{\delta\mathscr{L}}{\delta\phi_a}\,\delta\phi_a + \frac{\delta\mathscr{L}}{\delta\partial_\mu\phi_a}\,\delta\partial_\mu\phi_a\,. \tag{2}$$

Exchange $\delta\partial_\mu = \partial_\mu\delta$ in the second term and use then the Lagrange equations, $\delta\mathscr{L}/\delta\phi_a = \partial_\mu(\delta\mathscr{L}/\delta\partial_\mu\phi_a)$, in the first one. Combining the two terms using the product rule gives

$$0 = \delta\mathscr{L} = \partial_\mu\left(\frac{\delta\mathscr{L}}{\delta\partial_\mu\phi_a}\right)\delta\phi_a + \frac{\delta\mathscr{L}}{\delta\partial_\mu\phi_a}\,\partial_\mu\delta\phi_a = \partial_\mu\left(\frac{\delta\mathscr{L}}{\delta\partial_\mu\phi_a}\,\delta\phi_a\right)\,. \tag{3}$$

Hence the invariance of $\mathscr{L}$ under the change $\delta\phi_a$ implies the existence of a conserved current, $\partial_\mu j^\mu = 0$, with

$$j^\mu = \frac{\delta\mathscr{L}}{\delta\partial_\mu\phi_a}\,\delta\phi_a\,. \tag{4}$$

If the transformation $\delta\phi_a$ leads to change in $\mathscr{L}$ that is a total four-divergence, $\delta\mathscr{L} = \partial_\mu K^\mu$, then the equations of motion remain invariant. The conserved current j^μ is changed to

$$j^\mu = \frac{\delta\mathscr{L}}{\delta\partial_\mu\phi_a}\,\delta\phi_a - K^\mu\,. \tag{5}$$

2. A complex scalar field.

Consider a complex, scalar field ϕ with mass m and a quartic self-interaction proportional to λ in $d = 4$ space-time dimensions.

a.) Write down its Lagrange density $\mathscr{L}_s$, explain your choice of signs and pre-factors (when physically relevant). (6 pts)

b.) Determine the mass dimension of all quantities in the Lagrange density $\mathscr{L}_s$. (6 pts)
c.) Show that the Lagrange density $\mathscr{L}_s$ is invariant under global phase transformations and determine the conserved current j^μ. (6 pts)

a.) It is easiest to start from two real fields ϕ_1 and ϕ_2, combining them afterwards into the complex field $\phi = (\phi_1 + \mathrm{i}\phi_2)/\sqrt{2}$.
We have to decide which signature we use for the metric, and choose $(+,-,-,-)$. A Lagrange function has the form $L = T - V$, and thus $\dot{\phi}^2$ should have a positive coefficient, while all other terms are negative. Thus we choose the Lagrange density as

$$\mathscr{L}_i = A(\dot{\phi}_i^2 - (\nabla\phi_i)^2) - Bm^2\phi_i^2 - C\frac{\lambda}{4!}\phi_i^4 \tag{6}$$

with $A, B, C > 0$ (Lorentz invariance requires that the coefficient of $\dot{\phi}^2$ and $(\nabla\phi)^2$ agree). This choice of signs can be confirmed by calculating the Hamiltonian density $\mathscr{H}$, and requiring that it is bounded from below and stable against small perturbations. The correct dispersion relation for a free particle requires $A = B$. The kinetic energy of a canonically normalised field has the coefficient $A = 1/2$; this gives the correct size of vacuum fluctuations. The choice of $C > 0$ is arbitrary. Expressed by the complex fields, the Lagrangian becomes

$$\mathscr{L}_s = \partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2\,. \tag{7}$$

(Alternatively, we could have used that unitarity requires that $\mathscr{L}$ is real. Thus we should use bilinear quantities $\phi^*\phi$ or $\partial_\mu\phi^*\partial^\mu\phi$.)

b.) The action $S = \int \mathrm{d}x\mathscr{L}$ enters as $\exp(\mathrm{i}/\hbar S)$ the path integral and is therefore in natural units dimensionless. Thus $\mathscr{L}$ has mass dimension 4. From the kinetic term, we see that the mass dimension of the scalar field is 1. Thus the mass dimension of m is, not surprisingly, 1, and λ is dimensionless.

c.) Under the combined global phase transformations $\phi \to \mathrm{e}^{\mathrm{i}\vartheta}\phi$ and $\phi^\dagger \to \mathrm{e}^{-\mathrm{i}\vartheta}\phi^\dagger$, the Lagrangian $\mathscr{L}_s$ is clearly invariant. With $\delta\phi = \mathrm{i}\phi$, $\delta\phi^\dagger = -\mathrm{i}\phi^\dagger$, the conserved current follows as

$$j^\mu \propto \mathrm{i}\left[\phi^\dagger\partial^\mu\phi - (\partial^\mu\phi^\dagger)\phi\right]\,. \tag{8}$$

[We can drop the real parameter ϑ, but should keep the imaginary unit i such that the charge is real.]

3. Scalar QED.
Consider now the complex, scalar field ϕ coupled to the photon A^μ, i.e. a massless spin-1 field which field-strength satisfies $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.
a.) Find the coupling $\mathscr{L}_I$ between ϕ and A^μ requiring that $\mathscr{L}_s + \mathscr{L}_I$ is invariant under local phase transformations; determine the transformation law for A_μ. (10 pts)
b.) Write down the generating functionals for disconnected and connected Green functions of this theory. (6 pts)

c.) How does one obtain connected Green functions for the photon from the generating functional? (4 pts)
d.) Find the Feynman rules for the vertices involving photons and scalars. (6 pts)
e.) Define the superficial degree of divergence D and draw for each of the cases $D = \{0, 1, 2\}$ one 1-loop Feynman diagram. (7 pts)

a.) In case of global transformations, $\phi(x) \to \psi'(x) = U\phi(x)$, the normal derivative transformed as $\partial_\mu\phi(x) \to [\partial_\mu\phi(x)]' = U[\partial_\mu\psi(x)]$ and thus the kinetic energy was invariant. We define a new covariant derivative D_μ requiring

$$D_\mu\phi(x) \to [D_\mu\phi(x)]' = U(x)[D_\mu\phi(x)] \tag{9}$$

such that $(D_\mu\phi')^\dagger(D^\mu\phi') = (D_\mu\phi)^\dagger U^\dagger U(D^\mu\phi) = (D_\mu\phi)^\dagger D^\mu\phi$. The gauge field should compensate the difference between the normal and the covariant derivative,

$$D_\mu\phi(x) = [\partial_\mu + \mathrm{i}qA_\mu(x)]\phi(x)\,. \tag{10}$$

Now we determine the transformation properties of D_μ and A_μ demanding that $\phi'(x) = U(x)\phi(x)$ and (9) hold. Combining both requirements gives

$$D_\mu\phi(x) \to [D_\mu\phi]' = UD_\mu\phi = UD_\mu U^{-1}U\phi = UD_\mu U^{-1}\phi'\,, \tag{11}$$

and thus the covariant derivative transforms as $D'_\mu = UD_\mu U^{-1}$. Using its definition (10), we find

$$[D_\mu\phi]' = [\partial_\mu + \mathrm{i}qA'_\mu]U\phi = UD_\mu\phi = U[\partial_\mu + \mathrm{i}qA_\mu]\phi\,. \tag{12}$$

[Although not necessary, we do not use that A_μ is abelian.] Compare the second and the fourth term, after having performed the differentiation $\partial_\mu(U\phi)$. The result

$$[(\partial_\mu U) + \mathrm{i}gA'_\mu U]\phi = \mathrm{i}gUA_\mu\phi \tag{13}$$

should be valid for arbitrary ϕ and hence we arrive after multiplying from the right with $U^\dagger$ at

$$A_\mu \to A'_\mu = UA_\mu U^\dagger + \frac{\mathrm{i}}{g}(\partial_\mu U)U^\dagger = A_\mu - \partial_\mu\vartheta$$

where the last step is valid for an abelian transformation as in the case of a photon.
We obtain $\mathscr{L}_I$ by multiplying out the covariant derivatives,

$$(D_\mu\phi)^\dagger D^\mu\phi = (\partial_\mu\phi)^\dagger\partial^\mu\phi - \mathrm{i}q\left[\phi^\dagger\partial^\mu\phi - (\partial^\mu\phi^\dagger)\phi\right]A_\mu + q^2\phi^\dagger\phi A_\mu A^\mu = (\partial_\mu\phi)^\dagger\partial^\mu\phi + \mathscr{L}_I\,.$$

[If you use as start the ansatz $\mathscr{L}_I = qA_\mu j^\mu$, you have to realise that the replacement $\partial_\mu \to D_\mu$ is necessary in the Noether current.]

b.) The complete classical Lagrangian is

$$\mathscr{L}_{\mathrm{cl}} = \mathscr{L}_s + \mathscr{L}_I - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,.$$

To make the path-integral in a covariant gauge well-defined for an abelian gauge field, it is sufficient to add a gauge-fixing term

$$\mathscr{L}_{\rm gf} = -\frac{1}{2\xi}(\partial^\mu A_\mu)^2 \,.$$

Finaly, we add to $\mathscr{L}_{\rm eff}$ sources J_μ coupled linearly to the fields,

$$\mathscr{L}_{\rm eff} = \mathscr{L}_{\rm cl} + \mathscr{L}_{\rm gf} + J^\dagger\phi + \phi^\dagger J + J^\mu A_\mu \,. \tag{14}$$

The generating functional Z for disconnected Green functions is the path integral over fields over $\exp(\mathrm{i}\int d^4x\mathscr{L}_{\rm eff}\})$,

$$Z[J,J^*,J^\mu] = \int \mathcal{D}\phi\mathcal{D}\phi^*\mathcal{D}A_\mu \, \exp\{\mathrm{i}\int d^4x\mathscr{L}_{\rm eff}\} = \mathrm{e}^{\mathrm{i}W[J,J^*,J^\mu]}. \tag{15}$$

[We assume impicitely the Feynman prescription to ensure the convergence of the path integral.]

c.) $W[J,J^*,J^\mu]$ generates connected Green functions for the photon via

$$\frac{1}{\mathrm{i}^n}\frac{\mathrm{i}\delta^n W}{\delta J_\mu(x_1)\cdots\delta J_\nu(x_n)}\bigg)\bigg|_{J=0} = G_{\mu\cdots\nu}(x_1,\ldots,x_n)\,. \tag{16}$$

d.) We go to momentum space performing a Fourier transformation of $\mathscr{L}_I$. For the trilinear vertex,

$$\begin{aligned} F &= -\mathrm{i}q\int \mathrm{d}^4p_1\mathrm{d}^4p_2\mathrm{d}^4p_3\,(2\pi)^4\delta(p_1-p_2-p_3)\left[\phi^\dagger(p_1)\partial^\mu\phi(p_2) - (\partial^\mu\phi^\dagger(p_1)\phi(p_2))\right]A_\mu(p_3) \quad (17)\\ &= -\mathrm{i}q(-\mathrm{i}p_2^\mu - \mathrm{i}p_1^\mu)\phi^\dagger(p_1)\phi(p_2)A_\mu(p_3) \end{aligned}$$

Multiplying by i and taken derivatives w.r.t. to the three fields, we can read of as rule for the $\phi^\dagger\phi A_\mu$ vertex in case of 2 ϕ particles $-\mathrm{i}q(p_1^\mu + p_2^\mu)$. Since antiparticles propagate backwards, their momenta eneters with a minus sign. Proceeding similarly for the quartic vertex, $\phi^\dagger\phi A_\mu A^\mu$, we find $2\mathrm{i}q^2\eta_{\mu\nu}$.

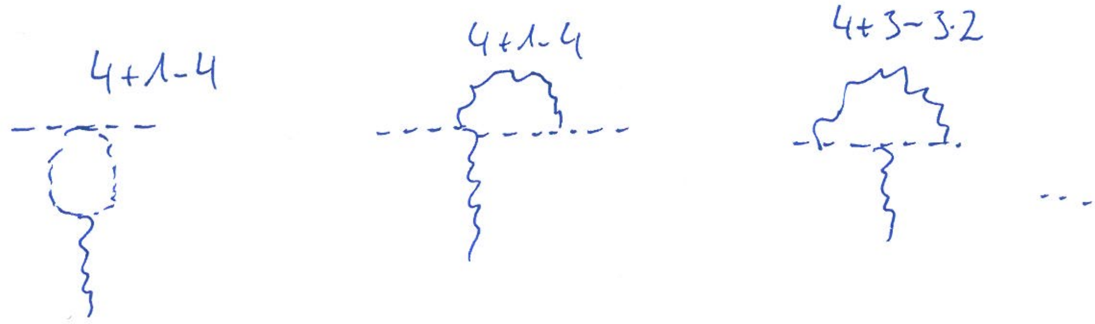
e.) The superficial degree D of divergence of a 1PI Feynman graph is given by the number of momenta in the numerator minus the number of momenta in the denominator. In $d=4$ space-time dimensions, L independent loop momenta contribute thus $4L$ and derivative couplings a factor d_V counting the power of derivatives from all vertices, while I internal bosonic lines subtract the factor $2I$. Combined

$$D = 4L + d_v - 2I.$$

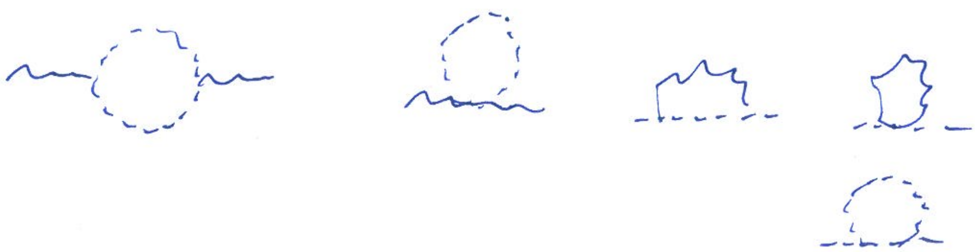
where L is the number of independent loop momenta and I the number of internal lines. Examples are:

- $D = 0$

- $D = 1$



- $D = 2$



4. Tensor decomposition.
Consider the *real* decay process $\mu \to e+\gamma$ in Minkowski space, allowing for parity violation. Write its Lorentz invariant amplitude $\mathcal{A}$ as $\mathcal{A}(\mu \to e+\gamma) = \varepsilon_\lambda \langle e| J^\lambda_{\rm em} |\mu\rangle$ where $J^\lambda_{\rm em}$ denotes the electromagnetic current and decompose it in scalar functions $A, B, \ldots$ as

$$\langle e| J^\lambda_{\rm em} |\mu\rangle = \bar{u}_e(p')[A\gamma^\lambda + \ldots]u_\mu(p) \,.$$

Use the symmetries to express $\mathcal{A}$ by the minimal number of scalar functions required. [Note the difference to the treatment of the electromagnetic vertex in the lectures where the photon was virtual.] (10 pts)

Translation invariance of Minkowski space implies four-momentum conservation, i.e. $q = p' - p$. Translation invariance combined with the on-shell condition means that the photon sees only the momentum difference $q = p' - p$, not the individual momenta p and p'. Hence $J^\lambda_{\rm em}$ can be only a function of the momentum difference, $J^\lambda_{\rm em} = J^\lambda_{\rm em}(q^\mu)$, while the arbitrary scalar functions can depend only on q^2.
We have to form all possible vectors out of the momenta q_μ and the 16 basis elements

$$\Gamma_i = \left\{1, \gamma^\mu, \gamma^\mu\gamma^5, \sigma^{\mu\nu}, \gamma^5\right\}$$

of the Clifford algebra.

$$J^\mu_{\rm em}(q) = (A + B\gamma^5)\gamma^\mu + (C + D\gamma^5)q^\mu + (D + E\gamma^5)\mathrm{i}\sigma^{\mu\nu}q_\nu \,.$$

The electromagnetic current $J^\lambda_{\rm em}$ is conserved, $\partial_\lambda J^\lambda_{\rm em}(x) = 0$ or $q_\lambda J^\lambda_{\rm em}(q) = 0$. This gives the condition

$$-m_e(A + B\gamma^5) + m_\mu(A - B\gamma^5) + q^2(C + D\gamma^5) = 0$$

or

$$A = B = 0$$

as the photon is on-shell, $q^2 = 0$. Finally we use that the photon is transverse, $\varepsilon^\mu q_\mu = 0$, to obtain

$$\mathcal{A}(\mu \to e + \gamma) = \varepsilon_\lambda \langle e| J^\lambda_{\rm em} |\mu\rangle = \varepsilon_\lambda \bar{u}_e(p')[(D + E\gamma^5){\rm i}\sigma^{\lambda\nu} q_\nu] u_\mu(p)\,.$$

Thus only the on-shell values of the magnetic form factors, $D(q^2 = 0)$ and $E(q^2 = 0)$, contribute to the decay process.

Some formulas

The Gamma matrices satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{18}$$

and are in the Weyl or chiral representation given by

$$\gamma^0 = 1 \otimes \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{19}$$

$$\gamma^i = \sigma^i \otimes {\rm i}\tau_3 = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{20}$$

$$\gamma^5 = 1 \otimes \tau_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{21}$$

$$\psi_L = \frac{1}{2}(1 - \gamma^5)\psi \equiv P_L\psi \quad \text{and} \quad \psi_R = \frac{1}{2}(1 + \gamma^5)\psi \equiv P_R\psi\,. \tag{22}$$

$$\sigma^{\mu\nu} = \frac{\rm i}{2}[\gamma^\mu, \gamma^\nu] \tag{23}$$