NTNU Trondheim, Institutt for fysikk

Examination for FY3464 Quantum Field Theory I

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Allowed tools: mathematical tables

1. Spin zero.

Consider a real, scalar field ϕ with mass m and a quartic self-interaction proportional to λ in $d = 4$ space-time dimensions. (4 pts)

a.) Write down the Lagrange density $\mathscr{L}$, explain your choice of signs and pre-factors (when physically relevant). (4 pts)

b.) Determine the mass dimension of all quantities in the Lagrange density $\mathscr{L}$. (6 pts)

c.) Draw the Feynman diagrams for $\phi\phi \to \phi\phi$ scattering at $\mathcal{O}(\lambda^2)$, determine the symmetry factor of these diagrams, and write down the expression for the Feynman amplitude i$\mathcal{A}$ of this process in momentum space. (8 pts)

d.) The one loop correction to the scalar propagator is

$$G^{(2)}(p) = \frac{\mathrm{i}}{p^2 - m^2 - \frac{\mathrm{i}\lambda}{2}\Delta_F(0) + \mathrm{i}\varepsilon}\,. \tag{1}$$

Calculate the self-energy or mass correction $\delta m^2 = \frac{\mathrm{i}\lambda}{2}\Delta_F(0)$ in dimensional regularisation (DR). You should end up with something of the form (10 pts)

$$\delta m^2 = \lambda m^2[a/\varepsilon + b + c\ln(\mu^2/m^2)]\,. \tag{2}$$

e.) What is your interpretation of the dependence of δm^2 on the parameter μ in Eq. (2)? [max. 50 words or one formula without explicit calculation is enough] (4 pts)

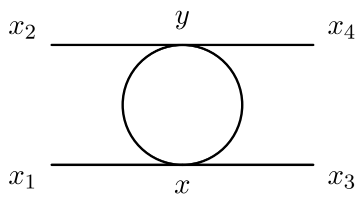
a. We have first to decide which signature we use for the metric, and choose $(+,-,-,-)$. A Lagrange function has the form $L = T - V$, and thus $\dot\phi^2$ should have a positive coefficient, while all other terms are negative. Thus we choose the Lagrange density as

$$\mathscr{L} = A(\dot\phi^2 - (\boldsymbol{\nabla}\phi)^2) - Bm^2\phi^2 - C\frac{\lambda}{4!}\phi^4 \tag{3}$$

with $A, B, C > 0$ (Lorentz invariance requires that the coefficient of $\dot\phi^2$ and $(\boldsymbol{\nabla}\phi)^2$ agree). This choice of signs can be confirmed by calculating the Hamiltonian density $\mathscr{H}$, and requiring that it is bounded from below and stable against small perturbations. The correct dispersion relation for a free particle requires $A = B$. The kinetic energy of a canonically normalised field has the coefficient $A = 1/2$; this gives the correct size of vacuum fluctuations and is, e.g., assumed in the standard form of propagators. The choice of C is arbitrary; other choices are compensated by a corresponding change in the symmetry factor of Feynman diagrams. we set $C = 1$.

b. The action $S = \int \mathrm{d}x \mathscr{L}$ enters as $\exp(\mathrm{i}/\hbar S)$ the path integral and is therefore in natural units dimensionless. Thus $\mathscr{L}$ has mass dimension 4. From the kinetic term, we see that the mass dimension of the scalar field is 1. Thus the mass dimension of m is, not surprisingly, 1, and λ is dimensionless.

c. In coordinate space, we have to connect four external points (say $x_1, \ldots, x_4$) with the help of two vertices (say at x and y) which combine each four lines. An example is



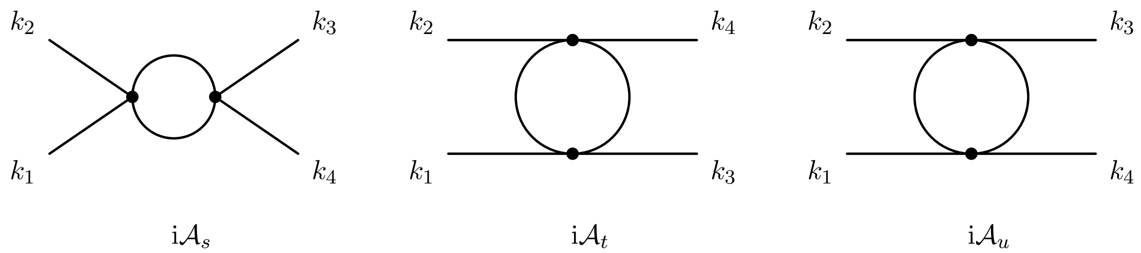
Two other diagrams are obtained connecting x_1 with x_2 or x_4. In order to determine the symmetry factor, we consider the expression for the four-point function corresponding to the graph shown above,

$$\frac{1}{2!}\left(-\mathrm{i}\frac{\lambda}{4!}\right)^2 \int \mathrm{d}^4x\mathrm{d}^4y\langle 0|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi^4(x)\phi^4(y)\}|0\rangle + (x \leftrightarrow y)\,, \tag{4}$$

and count the number of possible contractions: We can connect $\phi(x_1)$ with each one of the four $\phi(x)$, and then $\phi(x_3)$ with one of the three remaining $\phi(x)$. This gives 4×3 possibilities. Another 4×3 possibilities come by the same reasoning from the upper part of the graph. The remaining pairs $\phi^2(x)$ and $\phi^2(y)$ can be combined in two possibilities. Finally, the factor $1/2!$ from the Taylor expansion is canceled by the exchange graph. Thus the symmetry factor is

$$S = \frac{1}{2!}2!\left(\frac{4\times 3}{4!}\right)^2 2 = \frac{1}{2}\,. \tag{5}$$

Next we determine the Feynman amplitude in momentum space. We associate mathematical expressions to the symbols of the following graphs



as follows: We replace internal propagators by $\mathrm{i}\Delta(k)$, external lines by 1 and vertices by $-\mathrm{i}\lambda$. Imposing four-momentum conservation at the two vertices leaves one free loop momentum, which we call p. The momentum of the other propagator is then fixed to $p-q$, where $q^2 = s = (p_1+p_2)^2$,

$q^2 = t = (p_1 - p_3)^2$, and $q^2 = u = (p_1 - p_4)^2$ for the three graphs shown. Thus the Feynman amplitude at order $\mathcal{O}(\lambda^2)$ is $\mathrm{i}\mathcal{A}^{(2)} = \mathrm{i}\mathcal{A}_s^{(2)} + \mathrm{i}\mathcal{A}_t^{(2)} + \mathrm{i}\mathcal{A}_u^{(2)}$ with

$$\mathrm{i}\mathcal{A}_q^{(2)} = \frac{1}{2}\lambda^2 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{1}{[p^2 - m^2 + \mathrm{i}\varepsilon]} \frac{1}{[(p-q)^2 - m^2 + \mathrm{i}\varepsilon]} \,. \tag{6}$$

d. We add the mass scale μ^{4-n} and perform a Wick rotation,

$$\mu^{4-n}\mathrm{i}\Delta_F(0) = \int \frac{\mathrm{d}^4 k^2}{(2\pi)^4} \frac{1}{k^2 + m^2} \,. \tag{7}$$

Next we use the Schwinger's proper-time representation,

$$\int_0^\infty \mathrm{d}s \int \frac{\mathrm{d}^n k}{(2\pi)^n} \,\mathrm{e}^{-s(k^2+m^2)} = \frac{1}{(4\pi)^{n/2}} \int_0^\infty \mathrm{d}s\, s^{-n/2} \mathrm{e}^{-sm^2} = \frac{(m^2)^{\frac{n}{2}-1}}{(4\pi)^{n/2}} \Gamma\left(1 - \frac{n}{2}\right) \,. \tag{8}$$

where the substitution $x = sm^2$ transformed the integral into one of the standard representations of the gamma function. Now we expand

$$\delta m^2 = \lambda \mu^{4-n}\mathrm{i}\Delta_F(0) = \lambda \,\frac{m^2}{(4\pi)^2} \left(\frac{4\pi\mu^2}{m^2}\right)^{2-n/2} \Gamma(1 - n/2) \,. \tag{9}$$

in a Laurent series, separating pole terms in ε and a finite remainder using

$$\Gamma(1 - n/2) = \Gamma(-1 + \varepsilon/2) = -\frac{2}{\varepsilon} - 1 + \gamma + \mathcal{O}(\varepsilon) \tag{10}$$

and

$$a^{-\varepsilon/2} = \mathrm{e}^{-(\varepsilon/2)\ln a} = 1 - \frac{\varepsilon}{2} \ln a + \mathcal{O}(\varepsilon^2) \,. \tag{11}$$

Thus the mass correction is given by

$$\lambda \mu^{4-n}\mathrm{i}\Delta_F(0) \propto m^2 \left[-\frac{2}{\varepsilon} - 1 + \gamma + \mathcal{O}(\varepsilon)\right] \left[1 + \frac{\varepsilon}{2} \ln\left(\frac{4\pi\mu^2}{m^2}\right) + \mathcal{O}(\varepsilon^2)\right] \,. \tag{12}$$

$$= m^2 \left[-\frac{2}{\varepsilon} - 1 + \gamma - \ln\left(\frac{4\pi\mu^2}{m^2}\right) + \mathcal{O}(\varepsilon)\right] \,. \tag{13}$$

(Note that the result is still in Euclidean space, going back results in $m^2 \to -m^2$.)

e.) We still have to connect the quantity $m^2 + \delta m^2$ to the mass m_{phy} observed at a given scale Q^2. Performing this process (renormalisation), the scale μ will be replaced by the physical scale Q^2. Alternatively, we can use that amplitudes or Green functions like G^2 should be independent of μ; this will convert parameters like the mass m_{phy} into a scale dependent, running mass (if we perform a calculation at finite order perturbation theory).

2. Spin one-half. Consider a theory of two Weyl fields, a left-chiral field ϕ_L and a right-chiral field ϕ_R, with kinetic energy

$$\mathscr{L}_0 = \mathrm{i}\phi_R^\dagger \sigma^\mu \partial_\mu \phi_R + \mathrm{i}\phi_L^\dagger \bar{\sigma}^\mu \partial_\mu \phi_L \tag{14}$$

a.) Add a Dirac mass term $\mathscr{L}_D$. (3 pts)
b.) Find the transformation property of $\mathscr{L}_0$ and $\mathscr{L}_D$ under parity, $P\boldsymbol{x} = -\boldsymbol{x}$. (4 pts)
c.) Add a coupling $\mathscr{L}_{\text{int}}$ to the photon A^μ such that the coupling constant is dimensionless. (4 pts)

a. The Dirac mass term expressed by Weyl fields is

$$\mathscr{L} = -m(\phi_L^\dagger \phi_R + \phi_R^\dagger \phi_L)\,, \tag{15}$$

as follows e.g. from

$$\bar\psi\psi = \bar\psi\left(P_L^2 + P_R^2\right)\psi = \psi^\dagger\left(P_R\gamma^0 P_L + P_L\gamma^0 P_R\right)\psi = \bar\psi_R\psi_L + \bar\psi_L\psi_R\,. \tag{16}$$

b. $P\boldsymbol{x} = -\boldsymbol{x}$ implies $P\boldsymbol{\nabla} = -\boldsymbol{\nabla}$. Using the definitions $\sigma^\mu = (1, \boldsymbol{\sigma})$ and $\bar\sigma^\mu = (1, -\boldsymbol{\sigma})$, we have $P(\sigma^\mu\partial_\mu) = \bar\sigma^\mu\partial_\mu$ and $P(\bar\sigma^\mu\partial_\mu) = \sigma^\mu\partial_\mu$. Combined with $P\phi_L = \phi_R$ and $P\phi_R = \phi_L$, we see that parity exchanges the first and the second term in $\mathscr{L}_0$. The same holds for the Dirac mass term. Thus the combination of a left-chiral field and a right-chiral field Weyl field in $\mathscr{L}_o$ is invariant under parity, as well as a Dirac mass term.

c. From the kinetic energy of the Weyl fields, we find that the fermion fields have mass dimension 3/2. From the Maxwell Lagrangian given below, we see that the photon field (as any bosonic field) has dimension 1. Thus the two terms $\phi_R^\dagger\sigma^\mu\phi_R$ and $\phi_L^\dagger\bar\sigma^\mu\phi_L$ transform as (pseudo-) vectors and have dimension 3. The interaction

$$q(\phi_R^\dagger\sigma^\mu\phi_R + \phi_L^\dagger\bar\sigma^\mu\phi_L)A_\mu \tag{17}$$

has thus a dimensionless coupling q; it transforms as a scalar is thus a suitable interaction term $\mathscr{L}_{\text{int}}$.

3. Spin one.
Consider a massless spin-one particle, e.g. the photon A^μ with Lagrange density

$$\mathscr{L}_{\text{eff}} = \mathscr{L}_{\text{cl}} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2 = -\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2\,. \tag{18}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.
a.) List the symmetries of $\mathscr{L}_{\text{cl}}$, and of $\mathscr{L}_{\text{eff}}$. (5 pts)
b.) Derive the corresponding propagator $D_{\mu\nu}(k)$. [You don't have to care how the pole is handled.] (10 pts)
c.) Write down the generating functionals for disconnected and connected Green functions of this theory. (4 pts)
d.) How does one obtain connected Green functions from the generating functional? (3 pts)

e.) What are the two main changes in $\mathscr{L}_{\rm cl}$ and in $\mathscr{L}_{\rm eff} - \mathscr{L}_{\rm cl}$ in case of a non-abelian theory? [max. 50 words] (4 pts)

a. Continuous space-time symmetries: $\mathscr{L}_{\rm eff}$ is invariant under Lorentz transformations (3 boosts, 3 rotations) and translations (4). It contains no mass parameter and is thus conformally invariant (1 scale and 4 special conformal transformations). Internal symmetries: $\mathscr{L}_{\rm cl}$ is invariant under local gauge transformations, $A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \partial_\mu \Lambda(x)$. The local gauge invariance is broken by the gauge fixing term, which however respects global gauge transformations $A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \partial_\mu \Lambda$. (Otherwise current conservation would be broken by $\mathscr{L}_{gf}$.) And there are still discrete symmetries...

b. Step 1: massaging the Maxwell part into standard form,

$$\begin{aligned}\mathscr{L}_{\rm cl} &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2} \left(\partial_\mu A^\nu \partial^\mu A_\nu - \partial^\nu A_\mu \partial^\mu A_\nu \right) \\ &= \frac{1}{2} \left(A^\nu \partial_\mu \partial^\mu A_\nu - A_\mu \partial^\mu \partial^\nu A_\nu \right) = \frac{1}{2} A_\mu \left[\eta^{\mu\nu} \Box - \partial^\mu \partial^\nu \right] A_\nu = \frac{1}{2} A^\nu D^{-1}_{\mu\nu} A^\mu \,,\end{aligned}$$

Performing a Fourier transformation and Combining with the gauge-fixing part gives

$$P^{\mu\nu} = -k^2 \eta^{\mu\nu} + (1 - \xi^{-1}) k^\mu k^\nu \,. \tag{19}$$

Now we use the tensor method, either splitting the this expression into $\eta^{\mu\nu}$ and $k^\mu k^\nu$, or into its transverse and a longitudinal parts,

$$\begin{aligned} P^{\mu\nu} &= -k^2 \left(P_T^{\mu\nu} + \frac{k^\mu k^\nu}{k^2} \right) + (1 - \xi^{-1}) k^\mu k^\nu \\ &= -k^2 P_T^{\mu\nu} - \xi^{-1} k^2 P_L^{\mu\nu} \,. \end{aligned} \tag{20}$$

Since $P_T^{\mu\nu}$ and $P_L^{\mu\nu}$ project on orthogonal sub-spaces, we obtain the inverse $P^{-1}_{\mu\nu}$ simply by inverting their pre-factors. Thus the photon propagator in R_ξ gauge is given by

$$\mathrm{i} D_F^{\mu\nu}(k^2) = \frac{-\mathrm{i} P_T^{\mu\nu}}{k^2 + \mathrm{i}\varepsilon} + \frac{-\mathrm{i}\xi P_L^{\mu\nu}}{k^2 + \mathrm{i}\varepsilon} = \frac{-\mathrm{i}}{k^2 + \mathrm{i}\varepsilon} \left[\eta_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2 + \mathrm{i}\varepsilon} \right] . \tag{21}$$

c.) We add to $\mathscr{L}_{\rm eff}$ sources J_μ coupled linearly to the fields,

$$\mathscr{L}_{\rm eff} = \mathscr{L}_{\rm cl} + \mathscr{L}_{\rm gf} + J^\mu A_\mu \,. \tag{22}$$

The generating functional Z for disconnected Green functions is the path integral over fields over $\exp(\mathrm{i} \int d^4x \mathscr{L}_{\rm eff}\})$,

$$Z[J^\mu] = \int \mathcal{D}A \, \exp\{\mathrm{i} \int d^4x \mathscr{L}_{\rm eff}\} = \mathrm{e}^{\mathrm{i}W[J^\mu]}. \tag{23}$$

d.) $W[J^\mu]$ generates connected Green functions via

$$\frac{1}{\mathrm{i}^n} \frac{\mathrm{i}\delta^n W}{\delta J_\mu(x_1) \cdots \delta J_\nu(x_n)} \Bigg) \Bigg|_{J=0} = G_{\mu\cdots\nu}(x_1, \ldots, x_n) \,. \tag{24}$$

e. $\mathscr{L}_{\rm cl}$ contains now tri- and qudrilinear terms in the fields (with coefficient determined by the structure constants of the gauge group), i.e. the thoery is non-linear.
$\mathscr{L}_{\rm eff} - \mathscr{L}_{\rm cl}$ has to be modified either choosing a non-covariant gauge or adding a Fadeev-Popov ghost term.

Some formulas

The Pauli matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{25}$$

They satisfy $\sigma^i\sigma^j = \delta^{ij} + \mathrm{i}\varepsilon^{ijk}\sigma^k$. Combining the Pauli matrices with the unit matrix, we can construct the two 4-vectors $\sigma^\mu \equiv (1, \boldsymbol{\sigma})$ and $\bar{\sigma}^\mu \equiv (1, -\boldsymbol{\sigma})$.

The Gamma matrices satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{26}$$

and are in the Weyl or chiral representation given by

$$\gamma^0 = 1 \otimes \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{27}$$

$$\gamma^i = \sigma^i \otimes \mathrm{i}\tau_3 = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{28}$$

$$\gamma^5 = 1 \otimes \tau_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{29}$$

$$\psi_L = \frac{1}{2}(1-\gamma^5)\psi \equiv P_L\psi \quad \text{and} \quad \psi_R = \frac{1}{2}(1+\gamma^5)\psi \equiv P_R\psi\,. \tag{30}$$

$$\sigma^{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu] \tag{31}$$

$$\overline{\Gamma} = \gamma^0\Gamma^\dagger\gamma^0 \tag{32}$$

$$\frac{1}{ab} = \int_0^1 \frac{\mathrm{d}z}{[az + b(1-z)]^2}\,. \tag{33}$$

$$\frac{1}{k^2+m^2} = \int_0^\infty \mathrm{d}s\, \mathrm{e}^{-s(k^2+m^2)} \tag{34}$$

$$\int_{-\infty}^\infty \mathrm{d}x \exp(-x^2/2) = \sqrt{2\pi} \tag{35}$$

$$f^{-\varepsilon/2} = 1 - \frac{\varepsilon}{2}\ln f + \mathcal{O}(\varepsilon^2)\,. \tag{36}$$

$$\Gamma(z) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t} t^{z-1} \tag{37}$$

$$\Gamma(n+1) = n! \tag{38}$$

$$\Gamma(-n+\varepsilon) = \frac{(-1)^n}{n!}\left[\frac{1}{\varepsilon} + \psi_1(n+1) + \mathcal{O}(\varepsilon)\right]\,, \tag{39}$$

$$\psi_1(n+1) = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \gamma\,, \tag{40}$$