~~observing that the sign of $x^0 - y^0$ cannot change if x and y are timelike. Thus, the only way to avoid a contradiction and to make time-ordering well-defined even for spacelike events, is to demand that~~

$$\langle[\phi(x),\phi(y)]\rangle = [\phi(x),\phi(y)] = 0 \text{ if } (x-y)^2 < 0. \tag{1.107}$$

~~Physically, this ensures that we are preserving causality: first cause, then effect. Let us then check if it our quantized field $\phi(x)$ indeed satisfies $[\phi(x),\phi(y)] = 0$ for $(x-y)^2 < 0$.~~

~~Firstly, we note that $\langle[\phi(x),\phi(y)]\rangle$ is a Lorentz-scalar (invariant), as we remarked earlier. If it is zero in one frame, it is zero in all frames. We also know that for spacelike vectors, a frame exists where x and y occur simultaneously: $x^0 = y^0$. This does not violate the condition $(x-y)^2 < 0$. We derived earlier that:~~

$$\langle\phi(x)\phi(y)\rangle) = \int \frac{d^3k}{(2\pi)^3 2\omega(\boldsymbol{k})} \mathrm{e}^{-ik(x-y)} \tag{1.108}$$

~~for the scalar field. In the frame where $x^0 = y^0$, we then have~~

$$\langle[\phi(x),\phi(y)]\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega(\boldsymbol{k})} [\mathrm{e}^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} - \mathrm{e}^{-i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}] = 0 \tag{1.109}$$

~~due to the symmetry under $\boldsymbol{k} \to (-\boldsymbol{k})$, so causality is preserved.~~

~~The requirement of commutation goes back to the principle of QM where two measurements, for instance of $\phi(x)$ and $\phi(y)$, do not affect each other only if the corresponding operators commute. This must be the case for measurements made of $\phi(x)$ and $\phi(y)$ when $(x-y)^2 < 0$, since the spacetime positions of x and y are out of casual contact: no physical signal can travel the distance from x to y within their time separation.~~

~~However, this does not mean that $\langle\phi(x)\phi(y)\rangle = 0$ for spacelike separation. Let us evalute their Lorentz-invariant quantity in the frame where $x^0 = y^0$. We get~~

$$\begin{aligned}
\langle\phi(x)\phi(y)\rangle &= \int \frac{d^3k}{(2\pi)^3 2\omega(\boldsymbol{k})} \mathrm{e}^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} \\
&= \frac{1}{2(2\pi)^3} \int_0^\infty k^2 dk \int_{-1}^{1} d(\cos\theta) \int_0^{2\pi} d\phi \frac{\mathrm{e}^{ik\cos\theta|\boldsymbol{x}-\boldsymbol{y}|}}{\sqrt{k^2+m^2}} \\
&= \frac{1}{4\pi^2} \int_0^\infty \frac{kdk}{\sqrt{k^2+m^2}} \frac{\sin(k|\boldsymbol{x}-\boldsymbol{y}|)}{|\boldsymbol{x}-\boldsymbol{y}|} \\
&= \frac{1}{8\pi^2|\boldsymbol{x}-\boldsymbol{y}|^2} \int_{-\infty}^{\infty} \frac{zdz}{z^2+m^2|\boldsymbol{x}-\boldsymbol{y}|^2} \sin z,
\end{aligned} \tag{1.110}$$

~~where k above is not a 4-vector, but $k = |\boldsymbol{k}|$. In the last step, we used that the integrand is symmetric in k to expand its integration range and also changed variable to $k|\boldsymbol{x}-\boldsymbol{y}|$. Note that the integral can still be written in a Lorentz invariant waysince in our chosen frame $x^0 = y^0$, meaning that $|\boldsymbol{x}-\boldsymbol{y}|^2 = -(x-y)^2$. To do the last integral, observe that we have *branch points* located at $z = \pm im|x-y|$. To make this text somewhat self-contained, let us talk a bit about branch points and branch cuts since those will play a role when evaluating integrals later in this course.~~

Interlude: branch points and branch cuts

A branch point for a multivalued function in the complex plane is defined as follows: z_0 is a branch point for $f(z)$ if you go around a small circle centered at z_0 and end up with a different result for $f(z)$ than you started with. We will show some practical examples of this below. In some cases, $z_0 = \infty$ is a branch point for a function. We use here the notation $z_0 = \infty$ for complex infinity: a complex number with infinite magnitude and *undefined* argument. Do not be disheartened by the undefined argument: the same is also true for $z_0 = 0$, and that number is not so scary. In fact, $1/\infty = 0$ and $1/0 = \infty$ are reciprocals of each other. The undefined argument is highlighted here because the complex infinity discussed above is different from a directed infinity which has infinite magnitude and a well-defined complex argument, such as $\lim_{x\to o} 1/x$ where x is a positive, real number. We will show below how to test if $z_0 = \infty$ is a branch point.

Example branch point: let $f(z) = [z(z+1)]^{1/3}$. Here, z_0 and $z_0 = -1$ are strong candidates for branch points, which is clear when we set $z - z_0 = r\mathrm{e}^{\mathrm{i}\theta}$ in order to check the behavior of $f(z)$ when traversing a small circle centered at z_0 (meaning we advance θ from 0 to 2π). First, we try $z_0 = 0$ and obtain

$$f(z) = [r\mathrm{e}^{\mathrm{i}\theta}(r\mathrm{e}^{\mathrm{i}\theta} + 1)]^{1/3}. \tag{1.111}$$

The question is now: is this result invariant if we go from θ to $\theta + 2\pi$? The factor $[(r\mathrm{e}^{\mathrm{i}\theta} + 1)]^{1/3}$ is invariant, but the factor $[r\mathrm{e}^{\mathrm{i}\theta}]^{1/3} \propto \mathrm{e}^{\mathrm{i}\theta/3}$ is not. We conclude that z_0 is a branch point. Same procedure done for $z_0 = -1$ shows that it is also a branch point. Now what about $z_0 = \infty$? Now, to test if $z_0 = \infty$ is a branch point for a function, it can be easier to check if $\xi_0 = 1/z_0 = 0$ is a branch point for $g(\xi) \equiv f(z)$ where $\xi = 1/z$. So let us rewrite $f(z)$ to

$$g(\xi) = \Big[\frac{1}{\xi}\Big(\frac{1}{\xi} + 1\Big)\Big]^{1/3}. \tag{1.112}$$

Set $\xi = r\mathrm{e}^{\mathrm{i}\theta}$ to test if $\xi = 0$ is a branch point. If it is, then $z = \infty$ is too. We get

$$g(\xi) = \frac{1}{r\mathrm{e}^{\mathrm{i}\theta/3}}\Big(\frac{1}{r\mathrm{e}^{\mathrm{i}\theta}} + 1\Big)^{1/3}. \tag{1.113}$$

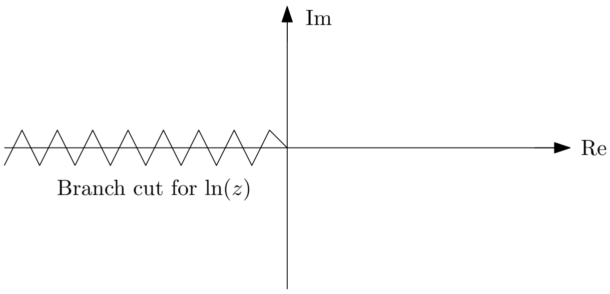
This is not invariant under $\theta \to \theta + 2\pi$, and we have thus found that $z_0 = \infty$.

Looking back on the last line of Eq. (1.110), we see that $z = \pm \mathrm{i}m|x-y|$ are branch points of the integrand. Some useful points to bear in mind regarding branch points:

- Single-valued functions do not have branch points: such points require multivaluedness.
- Branch points are, from the definition we gave, right next to multivalued points of the function. A function is therefore not analytic at a branch point (analytic = having a derivative and being single-valued in a neighborhoord around that point).
- A singularity is a where a function fails to be analytic $\to$ all branch points are singularities. However: all singularities are not branch points. For instance, $\sqrt{z}$ has one of its branch points at $z = 0$, which makes this point a singularity. But while $1/z$ also has a singularity at $z = 0$, it is not a branch point of $1/z$.
- There is no failproof strategy to find all branch points for a general function. One useful trick is to know which branch points basic functions like $\sqrt{z}$ have, and then try to recast complicated functions into basic ones which you know the branch points of.

Since branch points tells us that a function is multivalued at some point, it gives us a uniqueness problem regarding which value of the function we should use at specific points. The solution to this conundrum is the concept of *branch cuts*. This is a curve in the complex plane across which the principal value (or a specific branch) of an analytic, multivalued function is discontinuous. Again, we will give a concrete example of this below to see how it works. In other words, a branch cut is a curve in the complex plan which makes it possible to define a single analytic branch of a multivalued function, thus making it *single-valued* and removing the problematic ambiguity in the entire complex plane except on the curve itself. The analytic, multivalued function thus cannot be analytic on the branch cut itself since analyticity implies continuity. The function is instead singular on the entire branch cut after we have selected a specific branch of the function (such as its principal value).

Example branch cut: the complex logarithm is an example of a multivalued function in the complex plane with a branch cut. Writing $z = r\mathrm{e}^{\mathrm{i}\theta}$, we have $\ln(z) = \ln(r) + \mathrm{i}\theta$. But $\theta = \theta + 2\pi n$ gives the same complex number z, so $\ln(z)$ is multivalued due to $\mathrm{i}\theta \to \mathrm{i}\theta + 2\pi\mathrm{i}$. However, using a branch cut, we can make $\ln(z)$ single-valued in the entire complex plane except on a branch cut which extends from $z = 0$ to infinity in some direction. Note that $z = 0$ is a branch point of $\ln(z)$, and so branch cuts have to end on at least one branch point. Conventionally, the cut is taken as the negative real axis, shown in the figure.



Formally, we introduce the cut by defining ln(z) to be its principal value: the solution that has an imaginary part $\in (-\pi, \pi]$. We then have for $\epsilon \to 0^+$ that

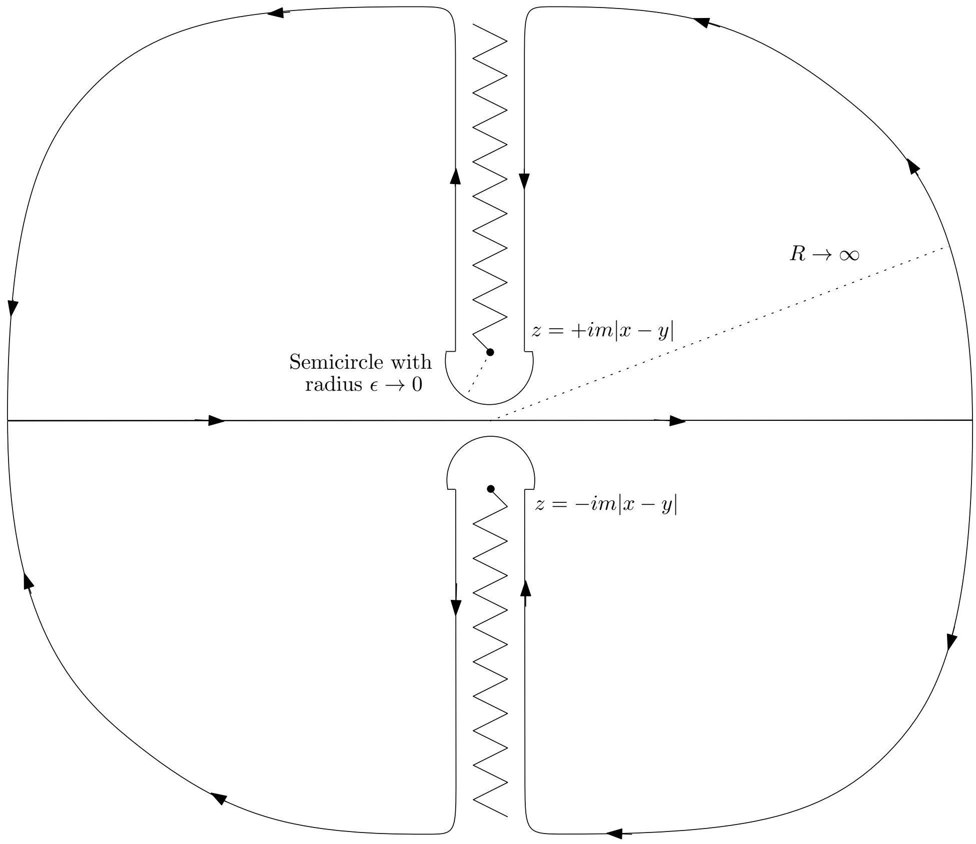
$$\ln(re^{\mathrm{i}(\pi-\epsilon)}) = \ln(r) + \mathrm{i}(\pi - \epsilon) \tag{1.114}$$

whereas

$$\ln(re^{\mathrm{i}(\pi+\epsilon)}) = \ln(r) + \mathrm{i}(-\pi + \epsilon). \tag{1.115}$$

It follows that with ln(z) has a jump of $2\pi\mathrm{i}$ across the branch cut. Note that $z = \infty$ is seen to be a branch point of ln(z) by writing $\ln(1/\xi) = -\ln(z)$. Note that when complex infinity is a branch point, as it is for ln(z), the branch cut from 0 to complex infinity can be taken in any direction.

Finally, we can return to Eq. (1.110) and evaluate it by using the carefully chosen contour shown in the figure.



Note that we could in principle have chosen the branch cuts to run between the two branch points, but then the integral we are interested in computing $\int_{-\infty}^{\infty}$ would have crossed the branch cut and could not be performed. Thus, for our purpose it is better to extend the cuts to infinity, since the integrand is discontinuous across the branch cuts and a branch cut has to end on at least one branch point.

~~Writing ...~~

~~I. Problems~~

1. ~~Prove that the invariance of $x^\mu x_\mu$ leads to the requirement $\eta_{\mu\nu} = \Lambda^\rho{}_\mu \eta_{\rho\sigma} \Lambda^\sigma{}_\nu$ for the Lorentz transformation tensor.~~

2. ~~Prove that $(\Lambda^{-1})^\sigma{}_\rho = \Lambda_\rho{}^\sigma$.~~

3. ~~Prove that $(\partial^2 + m^2) G_F(x-y) = -\mathrm{i}\delta^4(x-y)$. Hint: You can start by observing~~

$$
\begin{aligned}
(\partial^2 + m^2) G_F(x-y) = \int \frac{d^3k}{(2\pi)^3 2\omega(\boldsymbol{k})} \Bigg[&\partial_{x^0} \Big(\delta(x^0 - y^0) \mathrm{e}^{-\mathrm{i}k(x-y)} - \mathrm{i}\omega\theta(x^0 - y^0)\mathrm{e}^{-\mathrm{i}k(x-y)} \\
&- \delta(y^0 - x^0)\mathrm{e}^{\mathrm{i}k(x-y)} + \mathrm{i}\omega\theta(y^0 - x^0)\mathrm{e}^{\mathrm{i}k(x-y)} \Big) \\
&+ (\boldsymbol{k}^2 + m^2)[\theta(x^0 - y^0)\mathrm{e}^{-\mathrm{i}k(x-y)} + \theta(y^0 - x^0)\mathrm{e}^{\mathrm{i}k(x-y)}] \Bigg]
\end{aligned}
\tag{1.116}
$$

~~and then take it from there.~~