Instead, we have to use a relativistically invariant normalization:

$$\langle \vec{p}_1 | \vec{p}_2 \rangle = 2E_{p_1} (2\pi)^3 \delta^3(\vec{p}_1 - \vec{p}_2). \qquad (*)$$

Let us confirm that this is invariant.* Consider a boost in the p_z-direction (no loss of generality if final answer is independent of direction):

$$E_p' = \gamma(E_p - \beta p^z), \quad p^{z\prime} = \gamma(p^z - \beta E_p), \quad p^x = p^{x\prime}, \quad p^y = p^{y\prime}.$$

The normalization condition in the new frame is:

$$\langle \vec{p}_1{}' | \vec{p}_2{}' \rangle = 2E_{p_1}' (2\pi)^3 \delta^3(\vec{p}_1{}' - \vec{p}_2{}')$$

$$= 2\gamma\left(E_{p_1} - \beta p_1^z\right)(2\pi)^3 \delta\left[\gamma\left(p_1^z - \beta E_{p_1}\right) - p_2^{z\,\prime}\right]$$
$$\cdot\, \delta\left(p_1^{x\,\prime} - p_2^{x\,\prime}\right)\delta\left(p_1^{y\,\prime} - p_2^{y\,\prime}\right)$$

To simplify the first δ-function, we recall that the transformation of a δ-function for an argument vanishing at $x=0$ is:

$$\delta\left[g(x)\right] = \frac{1}{\left|\frac{dg}{dx}\right|_{x=0}} \cdot \delta(x)$$

Using this formula, we can rewrite the first δ-function as:

$$\langle \vec{p}_1^{\,\prime} | \vec{p}_2^{\,\prime} \rangle = 2\gamma\left(E_{p_1} - \beta p_1^z\right)(2\pi)^3 \delta\left(p_1^x - p_2^x\right)\delta\left(p_1^y - p_2^y\right)$$
$$\cdot\, \frac{1}{\gamma\left(1 - \beta \frac{dE_{p_1}}{dp_1^z}\right)} \delta\left(p_1^z - p_2^z\right)$$

Here, we used that $\frac{d\left[\gamma\left(p_1^z - \beta E_{p_1}\right) - \gamma\left(p_2^z - \beta E_{p_2}\right)\right]}{d\left(p_1^z - p_2^z\right)}$

$$= \gamma - \gamma\beta \frac{d\left(E_{p_1} - E_{p_2}\right)}{d\left(p_1^z - p_2^z\right)} = \gamma - \gamma\beta\left(\frac{p_1^z}{2E_{p_1}} + \frac{p_2^z}{2E_{p_2}}\right)$$

Is shown by using $E_{p_i} = \sqrt{\vec{p}_i^{\,2} + m_i^2}$ and introducing dummy variables $a = p_1^z + p_2^z$, $b = p_1^z - p_2^z$ so that $p_1^z = \frac{a+b}{2}$, $p_2^z = \frac{a-b}{2}$. Due to the $\delta(p_1^z - p_2^z)$, we can without loss of generality set $p_1^z = p_2^z$ and $E_{p_1} = E_{p_2}$, since everything is zero otherwise.

Evaluating the term in the second line, we get

$$1 - \beta \frac{dE_{P_1}}{dP_1^z} = 1 - \beta \frac{P_1^z}{\sqrt{\vec{P_1}^2 + m_1^2}} = \frac{E_{P_1} - \beta P_1^z}{E_{P_1}}$$

Putting everything together, we get:

$$\langle \vec{P_1}' | \vec{P_2}' \rangle = 2E_{P_1} (2\pi)^3 \delta^3(\vec{P_1} - \vec{P_2})$$

which proves the invariance of the normalization condition.

It is also useful to consider the Lorentz-invariance of integrals over momentum space, as such turn up e.g. in the context of completeness relations.

The integral $\int \frac{d^3p}{(2\pi)^3}$ is clearly not invariant, but

$\int \frac{d^4p}{(2\pi)^4} (2\pi) \delta(p^2 - m^2)$ consists only of invariant components.

Performing the integral over p^0:

$$\int dp^0 \delta(p^2 - m^2) = \int dp \, \delta[(p^0)^2 - \vec{p}^2 - m^2] = \frac{1}{2p^0} \Big|_{p^0 = E_p}$$

(used $\delta[g(x)] = \sum_i \frac{1}{|g'(x_i)|} \delta(x - x_i)$ where x_i are zeros of $g(x)$)

Then: $$\int \frac{d^4p}{(2\pi)^4} (2\pi) \delta(p^2 - m^2) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p}$$

is a relativistically invariant integral.

We can also show that the phase space volume element $\frac{d^3p}{E}$ itself is invariant. Consider a Lorentz-boost in the z-direction (any direction works).

$$p^x = p^{x\prime},\ p^y = p^{y\prime},\ p^{z\prime} = \gamma(p^z - \beta E),\ E' = \gamma(E - \beta p^z),$$

$\gamma = \frac{1}{\sqrt{1-\beta^2}}$, $\beta = \frac{v}{c}$. We have:

$$\frac{dp^{z\prime}}{dp^z} = \gamma\left(1 - \beta\frac{dE}{dp^z}\right) = \gamma\left(1 - \frac{\beta p^z}{E}\right)$$

Since $\frac{dE}{dp^z} = \frac{d}{dp^z}\sqrt{\vec{p}^{\,2} + m^2} = \frac{p^z}{E}$.

But now we have:

$$\frac{dp^{z\prime}}{dp^z} = \gamma\left(1 - \frac{\beta p^z}{E}\right) = \frac{\gamma(E - \beta p^z)}{E} = \frac{E'}{E}$$

$$\Rightarrow \frac{dp^{z\prime}}{E'} = \frac{dp^z}{E} \Rightarrow \underline{\frac{d^3p'}{E'} = \frac{d^3p}{E}}.$$

With the above treatment of relativistic normalization, we see that we can also construct a set of creation-annihilation operators $\tilde{a}, \tilde{a}^\dagger$:

$$\tilde{a}(\vec{p}) \equiv (2\pi)^{3/2}\sqrt{2E_p}\,a(\vec{p}),$$

$$\tilde{a}^\dagger(\vec{p}) \equiv (2\pi)^{3/2}\sqrt{2E_p}\,a^\dagger(\vec{p})$$

that have Lorentz invariant commutation relations:

$$[\tilde{a}(\vec{p}), \tilde{a}^+(\vec{p}\,')] = (2\pi)^3 (2E_p)\, \delta(\vec{p}-\vec{p}\,')$$

Finally, we then see that a Lorentz-invariant completeness relation must have the form:

$$\underline{1} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} |\vec{p}\rangle\langle\vec{p}|$$

for the momentum-states. Check:

$$1|\vec{k}\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} |\vec{p}\rangle\langle\vec{p}|\vec{k}\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} |\vec{p}\rangle\, 2E_p (2\pi)^3 \delta^3(\vec{p}-\vec{k})$$

$$= |\vec{k}\rangle.$$

Since $2E_p(2\pi)^3\delta^3(\vec{p}_1-\vec{p}_2) \sim eV \cdot eV^{-3} = eV^{-2}$, relativistically normalized states have dimension $|\vec{p}\rangle \sim eV^{-1}$.