$\mathcal{T}$ is called the T-matrix and contains information about scattering.

Due to the unitarity of S, we have

$$1 = (1 + i\mathcal{T})(1 - i\mathcal{T}^{\dagger}) \Rightarrow i\mathcal{T}\mathcal{T}^{\dagger} = \mathcal{T} - \mathcal{T}^{\dagger}.$$

In perturbation theory, one can neglect the l.h.s. since it is higher order in the interaction parameter λ than the r.h.s. Thus, to lowest order we have a Hermitian $\mathcal{T} = \mathcal{T}^{\dagger}$.

Note also that if the initial and final states are orthogonal,

$\langle \psi | \alpha \rangle = 0$, we have: $\langle \psi | S | \alpha \rangle = i \langle \psi | T | \alpha \rangle$.

Finally, we will actually not assume normalized states $|\psi\rangle$ and $|\alpha\rangle$, as this will turn out to be inconvenient later. Leaving them unnormalized, we can compute the probability for a transition from $|\alpha\rangle$ to $|\psi\rangle$ as follows.

We have $|f\rangle = \sum_i c_i |\psi_{i}, +T\rangle$, but since $|\psi_{i}, +T\rangle$ are not normalized (but still orthogonal), $|c_i|^2$ is <u>not</u> the prob. that the system is in state i.

Instead:

$\langle f | f \rangle = \sum_i |c_i|^2 \langle \psi_{i}, +T | \psi_{i}, +T \rangle$. We can write this as

$$1 = \sum_i P_i \quad \text{with} \quad P_i = \frac{|c_i|^2 \langle \psi_{i}, +T | \psi_{i}, +T \rangle}{\langle f | f \rangle}$$ and these can then be interpreted as proper probabilities.

Since $|f\rangle = e^{-2iHT}|\varphi\rangle$, we see that $\langle f|f\rangle = \langle\varphi|\varphi\rangle$.

The c_i- coeffs. themselves are found in the standard way:

$$c_i = \frac{\langle \psi_i, +T|f\rangle}{\langle\psi_i,+T|\psi_i,+T\rangle}.$$

This means that the probability for the system to end up in a particular state $|\psi_i,+T\rangle$ when it started in $|\varphi\rangle$ at $t=-T$ is:

$$P_{\varphi\psi} = \frac{|\langle\psi_i+T|f\rangle|^2}{\langle\psi_i+T|\psi_i,+T\rangle\langle\varphi|\varphi\rangle} = \frac{|\langle\psi|S|\varphi\rangle|^2}{\langle\psi\psi\rangle\langle\varphi|\varphi\rangle}$$

Since $|\psi_i,+T\rangle = e^{-iH_{\text{non-int}}\cdot T}|\psi\rangle$. Thus, if $\langle\psi|\varphi\rangle=0$ we get:

$$P_{\varphi\psi} = \frac{|\langle\psi|T|\varphi\rangle|^2}{\langle\psi|\psi\rangle\langle\varphi|\varphi\rangle}.$$

Without normalization, the correct completeness relation is also

$$\sum_i \frac{|\psi_i\rangle\langle\psi_i|}{\langle\psi_i|\psi_i\rangle} = 1.$$