$\int d\theta_1 d\theta_2\, \theta_1\theta_2 = -\int d\theta_2 d\theta_1 \theta_1\theta_2 = -\int d\theta_1 \theta_1 \int d\theta_2 \theta_2 = -1.$

As for the scalar field, the Gaussian integral will be of particular interest to us in the context of path integrals for fermions.

Consider first a complex Grassmann made up of two Grassmann variables : $\theta = \theta_1 + i\theta_2$ and $\bar{\theta} \equiv \theta_1 - i\theta_2$. Here, θ_1 and θ_2 are real G-variables : $\theta_i = \theta_i^*$. Note that generally $(\theta\eta)^* = \eta^*\theta^*$ for complex G-variables. The relation between differentials is

$$d\bar{\theta}\, d\theta = \det \begin{pmatrix} \frac{\partial\theta_1}{\partial\bar{\theta}} & \frac{\partial\theta_2}{\partial\bar{\theta}} \\ \frac{\partial\theta_1}{\partial\theta} & \frac{\partial\theta_2}{\partial\theta} \end{pmatrix} d\theta_1 d\theta_2 = \frac{-i}{2} d\theta_1 d\theta_2 \qquad (4)$$

The determinant appears on the opposite side of the eq. compared to c-number differentials. Let us briefly show why this is so.

Riemann integrals satisfy $\int dx \cdot f(ax) = \frac{1}{a}\int dx\, f(x)$ (inf. integration limits assumed / integration over all allowed values). Instead, Berezin integration satisfies:

$$\int d\theta\, f(a\theta) = a \int d\theta\, f(\theta)$$

The reason is that Berezin integration is equivalent to differentiation, as we remarked. To see this, let $\eta = a\theta$ be a new G-variable and consider that by definition, we have:

$$\int d\eta \cdot \eta = \int d\theta \cdot \theta = 1.$$

Thus: $\int d\eta \cdot \eta = \int d\eta \cdot a \cdot \theta = \int d\theta \cdot \theta$. Clearly, $a \cdot d\eta = d\theta$.

So $d\eta = \frac{1}{a} d\theta$ while in standard calculus we would have had

$d\eta = a\, d\theta$ (for $\eta, \theta \in \mathbb{C}$). This can now be generalized

to a transformation $\eta_i = \sum_j a_{ij}\, \theta_j$ (a_{ij} c-numbers)

$$\Rightarrow \det(a)\, d\eta_1 d\eta_2 \dots d\eta_N = d\theta_1 d\theta_2 \dots d\theta_N$$

with the $\det(a)$ on the opposite side of the eq. compared to the standard case.

Back to the Gaussian integral. Let b be a c-number. Then:

$$\int d\bar{\theta} d\theta\, e^{-\bar{\theta} b \theta} = \int d\bar{\theta} d\theta\, (1 - \bar{\theta} b \theta) = b.$$ | use that $(\bar{\theta} b \theta)^2 = 0$ in the expansion of $e^{-\bar{\theta} b \theta}$

Using (*) (previous page), we see that this is also equal to:

$$-\frac{i}{2}\int d\theta_1 d\theta_2\, e^{-2ib\,\theta_1\theta_2}$$

Now for N complex G-variables θ_j and $\bar{\theta}_j$, this Gaussian integral generalizes to:

$$\int \prod_{j=1}^{N} d\bar{\theta}_j d\theta_j \; e^{-\bar{\theta}_j B_{jk} \theta_k} = \int \prod_{j=1}^{N} d\bar{\theta}_j' d\theta_j' \; e^{-\sum_j \bar{\theta}_j' b_j \theta_j'}$$

$$= \prod_{j=1}^{N} b_j = \det(B). \qquad (+)$$

Here, we used a unitary transformation to diagonalize the matrix B_{jk}:

$U_{jk} B_{kl} U^{-1}_{lm} = b_j \delta_{jm}$ defined $\theta_j' = U_{jk}\theta_k$ [which gives

$\bar{\theta}_j' = U^*_{jk}\bar{\theta}_k = [(U^{-1})^T]_{jk}\bar{\theta}_k$ since $U^+ = (U^*)^T = U^{-1}$].

Since $\det[(U^{-1})^T] = \det(U^{-1})$, we then get that the measure is invariant:

$$\prod_{j=1}^{N} d\bar{\theta}_j d\theta_j = \underbrace{\det(U^{-1})\det(U)}_{=1} \prod_{j=1}^{N} d\bar{\theta}_j' d\theta_j' = \prod_{j=1}^{N} d\bar{\theta}_j' d\theta_j'$$

Observe how $(+)$ gives $\det(B)$ in the Grassmann case, while we saw that an ordinary Gaussian integral gives $\frac{1}{\det(B)}$.

Completing a square in a Grassmann Gaussian also works nicely.
For instance, consider the following integral with η and $\bar{\eta}$ coupled to $\bar{\theta}$ and θ:

$$\int d\bar{\theta}\, d\theta\, e^{-\bar{\theta} b \theta + \bar{\eta}\theta + \bar{\theta}\eta} = \int d\bar{\theta}\, d\theta \left(1 - \bar{\theta} b \theta + \bar{\eta}\theta\bar{\theta}\eta\right)$$

$$= b + \bar{\eta}\eta = b\, e^{\bar{\eta} b^{-1} \eta} \qquad (1)$$

However, we can also express the integral as:

$$\int d\bar{\theta}\, d\theta\, e^{-\bar{\theta} b \theta + \bar{\eta}\theta + \bar{\theta}\eta} = \int d\bar{\theta}\, d\theta\, e^{-(\bar{\theta} - \bar{\eta} b^{-1})\, b\, (\theta - b^{-1}\eta)}\, e^{\bar{\eta} b^{-1} \eta}$$

$$= e^{\bar{\eta} b^{-1} \eta} \underbrace{\int d\bar{\theta}\, d\theta\, e^{-(\bar{\theta} - \bar{\eta} b^{-1})\, b (\theta - b^{-1}\eta)}} = b\, e^{\bar{\eta} b^{-1} \eta} \qquad (2)$$

We know this is equal to b
from the invariance under constant shift
of variables

So completing the square in (2) gives the same result as direct evaluation in (1). Finally, we can define a δ- function in θ-variables, $\delta(\theta - \theta')$, such that

$$\int d\theta\, \delta(\theta - \theta')\, f(\theta) = f(\theta').$$

You can verify that this is satisfied by

$$\delta(\theta - \theta') = \theta - \theta'.$$