The next discrete symmetry is charge conjugation C.

As the name suggests, the C operation should reverse the sign of the charge we discussed previously so that $Q \to -Q$ and all values of $q \to -q$. In effect, C should exchange particles and antiparticles. Since $Q = \int d^3x\, j^0_V$, the transformation of the field $\psi(x)$ under C should be such that j^0_V changes sign while $\mathcal{L}$ is invariant.

Consider first a complex scalar field Lagrangian:

$\mathcal{L} = (\partial_\mu \varphi^*)(\partial^\mu \varphi) - m^2 \varphi^* \varphi$ is clearly invariant under $\varphi \to \varphi^*$.

The C-operation is simply complex conjugation for such a theory.

For a fermion field $\psi(x)$, it seems reasonable that complex conjugation should occur as well, but this is insufficient to make $\mathcal{L}$ invariant.

The mass-term of $\mathcal{L}_{\text{Dirac}}$ transforms under $\psi \to \psi^*$ as:

$$\underline{m\bar{\psi}\psi} \to m\psi^T \gamma^0 \psi^* = m\psi_\alpha \gamma^0_{\alpha\beta} \psi^*_\beta = -m\psi^*_\beta \gamma^0_{\alpha\beta} \psi_\alpha$$

$$= -m\psi^*_\alpha \gamma^0_{\beta\alpha} \psi_\beta \text{ (relabeled } \alpha \leftrightarrow \beta) = -m\psi^*_\alpha (\gamma^{0T})_{\alpha\beta} \psi_\beta = \underline{-m\bar{\psi}\psi}$$

where we used the anticommutativity of the components of the Dirac spinors.

We can fix this by modifying the transformation to $\psi \to i\gamma^2\psi^*$, but the kinetic term of $\mathcal{L} = i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi$ is still not invariant under this transformation. However - it only changes with a total divergence $\partial_\mu J^\mu$! In fact, the equivalent $\mathcal{L}'$ we

which means $\psi \to i\gamma^2\psi^*$ can still be regarded as a sym. trans. for $\mathcal{L}$ as the physics is unchanged.

mentioned previously:

$$\mathcal{L}' = \frac{i}{2}\bar\psi \partial\!\!\!/ \psi - \frac{i}{2}(\partial_\mu\bar\psi)\gamma^\mu\psi - m\bar\psi\psi$$

is invariant under $\psi \to i\gamma^2\psi^*$. The kinetic terms transform into each other. Let's show this for the first term explicitly.

We will use that:

- $\gamma^2 = -1$ • $\gamma^{0T} = \gamma^0$ • $\gamma^{\mu T}\gamma^0 = \begin{cases}\gamma^0\gamma^\mu & \text{if } \mu\neq 2\\ -\gamma^0\gamma^\mu & \text{if } \mu=2\end{cases}$
- $\{\gamma^2,\gamma^0\} = \{\gamma^2, \gamma^0\gamma^2\} = 0$ • $[\gamma^2,\gamma^0\gamma^1] = [\gamma^2,\gamma^0\gamma^3] = 0$.

Beginning:

$$\frac{i}{2}\bar\psi\partial\!\!\!/\psi \to \frac{i}{2}(i\gamma^2\psi)^\dagger\gamma^0\partial\!\!\!/(i\gamma^2\psi^*) = -\frac{i}{2}(\psi^T\gamma^{2T})\gamma^0\partial\!\!\!/\gamma^2\psi^*$$

$$= -\frac{i}{2}\psi_\alpha(\gamma^{2T}\gamma^0\partial\!\!\!/\gamma^2)_{\alpha\beta}\psi^*_\beta = \frac{i}{2}(\partial_\mu\psi^*_\beta)(\gamma^{2T}\gamma^0\gamma^\mu\gamma^2)_{\alpha\beta}\psi_\alpha \quad \Big|\ \text{anticom. of Dirac spinors}$$

$$= -\frac{i}{2}(\partial_\mu\psi^*_\alpha)(-\gamma^{2T}\gamma^0\gamma^\mu\gamma^2)_{\beta\alpha}\psi_\beta \quad \Big|\ \text{relabeled indices}$$

$$= -\frac{i}{2}(\partial_\mu\psi^*_\alpha)(-\gamma^{2T}\gamma^{\mu T}\gamma^{0T}\gamma^2)_{\alpha\beta}\psi_\beta$$

This is equal to the 2nd term in $\mathcal{L}'$ if $-\gamma^{2T}\gamma^{\mu T}\gamma^{0T}\gamma^2 = \gamma^0\gamma^\mu$. Checking:

$$-\gamma^{2T}\gamma^{\mu T}\gamma^{0T}\gamma^2 \overset{(\gamma^{2T}=+\gamma^2)}{=} -\gamma^2\gamma^{\mu T}\gamma^0\gamma^2 = \begin{cases} -\gamma^2\gamma^0\gamma^\mu\gamma^2 & \text{if } \mu\neq 2\\ +\gamma^2\gamma^0\gamma^\mu\gamma^2 & \text{if } \mu=2\end{cases}$$

$$= \begin{cases}\gamma^0\gamma^\mu & \text{if } \mu\neq 2\\ \gamma^0\gamma^\mu & \text{if } \mu=2\end{cases} = \gamma^0\gamma^\mu \ \square. \quad \Big|\ \text{used } (\gamma^2)^2 = -1$$

Note how C does not send $i \to (-i)$, it only transforms $\psi \to i\gamma^2\psi^*$. Note the relation $(i\gamma^2)^{-1}(\gamma^\mu)^*(i\gamma^2) = -\gamma^\mu$ which is useful to prove that $(i\gamma^2\psi^*)$ satisfies charge-reversed Dirac eq.

In a similar manner, one confirms that $j_V^\mu \to -j_V^\mu$ and $j_A^\mu \to j_A^\mu$ under C (by using $[\gamma^0\gamma^\mu, \gamma^5] = 0$ and $\gamma^{5T} = \gamma^5$).

Therefore, we have shown that C takes $Q \to -Q$ and thus swaps particles and antiparticles. Spins are not affected by C.

The charge conjugation symmetry guarantees the existence of an antiparticle fermion if the particle fermion exists. To see this, consider that the charge will enter the Dirac equation if we include a coupling to an EM field. More details later when discussing QED, but it shouldn't come as a surprise that $i\partial_\mu \to i\partial_\mu - eA_\mu$ for a particle with charge e (minimal substitution). Using the techniques for demonstrating C-invariance of $\mathcal{L}'$, you can verify that if ψ solves

$$[\gamma^\mu(i\partial_\mu - eA_\mu) - m]\psi = 0,$$

then $i\gamma^2\psi^* \equiv \psi^c$ solves the Dirac eq. with opposite charge:

$$[\gamma^\mu(i\partial_\mu + eA_\mu) - m]\psi^c = 0.$$

A chiral version of $\mathcal{L}'$ (involving e.g. just ψ_L like $\frac{i}{2}\bar{\psi}_L \partial\!\!\!/ \psi_L - \ldots$) is not invariant under C, as one obtains $\mathcal{L}' \to \frac{i}{2}\bar{\psi}_R \partial\!\!\!/ \psi_R - \ldots$.

But this is the same transformation as parity P accomplishes!

Thus, a chiral theories are invariant under the combination CP.