SCATTERING

Treating the computation of correlators via path integrals and Feynman diagrams is beautiful and rewarding in itself, giving rise to profound insights such as renormalisation, but we'll now show that the correlators we have discussed so extensively can be used for a practical purpose: predicting how particles scatter.

1 The S-matrix

A generic scattering problem assumes that there is an initial state in the far past with two widely separated particles, which scatter during some time period in the middle, and then evolves into a state in the far future where the particles (two <u>or more</u>) are again widely separated.

The time evolution of such a process is determined by the full interacting Hamiltonian H. Given an incoming state, we wish to know the probability amplitude of finding an outgoing state.

When the particles are widely separated, their interactions with each other can be neglected. However, the particles still have self-interactions (interacting with virtual surrounding fields) which, as we have seen, lead to mass-shifts and field-renormalisations (Z).

We call such particles "dressed". Assume now that there exists a Hamiltonian $H_{non\text{-}int}$ where the dressed particles do not interact with each other - we will soon demonstrate the existence of such a $H_{non\text{-}int}$. Let $|\alpha\rangle$ and $|\psi\rangle$ be two eigenstates of $H_{non\text{-}int}$ where each state corresponds to a set of dressed particles with fixed momenta and other internal quantum numbers [if these momenta and q.n. are different for $|\alpha\rangle$ and $|\psi\rangle$, we would have $\langle\psi|\alpha\rangle = 0$].

Consider now a state $|\Psi, -T\rangle$ at $t=-T$. It will evolve with the full interacting time-evolution operator e^{-iHt} to a state $|\Psi, +T\rangle$ in the future. Since we assume that the particles are widely separated at $-T$, $|\Psi, -T\rangle$ should equal one of the eigenstates $|\alpha\rangle$ of $H_{non\text{-}int}$: $|\Psi, -T\rangle = |\alpha\rangle$. This state will then evolve into some final state $|f\rangle$ at $t=T$: $|\Psi, +T\rangle = |f\rangle$. We then know that:

$$|f\rangle = e^{-iH\Delta t}|\alpha\rangle = e^{-2iHT}|\alpha\rangle.$$

At the same time, the state $|\Psi, +T\rangle$ in the far future should in general be a superposition of eigenstates of $H_{non\text{-}int}$:

$$|\Psi, +T\rangle = \sum_i c_i |\psi_i, +T\rangle$$

Since the particles are widely separated also at $+T$ and do not interact. Now, the relation between one of these eigenstates (non-interacting) $|\psi_i, +T\rangle$ and the same eigenstate in the past is governed by

$H_{non\text{-}int}$:

$$|\psi_i, +T\rangle = e^{-2iH_{non\text{-}int}T}\,|\psi_i\rangle .$$

(note how $|\alpha\rangle$ and $|\psi_i\rangle$ refer to Schrödinger pict. states at $t=-T$)

Look then at one particular of these eigenstates: $|\psi, +T\rangle = e^{-2iH_{non\text{-}int}T}|\psi\rangle$.

We can then write down the probability amplitude that the system has evolved to this particular state when it started in the state $|\alpha\rangle$:

$$\langle \psi, +T | f\rangle = \langle \psi | e^{2iH_{non\text{-}int}T} \cdot e^{-2iHT} |\alpha\rangle$$

since $|f\rangle$ is the general final state of the system.

It is conventional (and more symmetric) to express this in terms of non-interacting eigenstates at $t=0$. This can easily be done by evolving $|\alpha\rangle$ and $|\psi\rangle$ to $t=0$ with $H_{non\text{-}int}$:

$$|\alpha\rangle_0 = e^{-iH_{non\text{-}int}T}|\alpha\rangle \quad (\alpha \to \psi \text{ as well}).$$

We then get:

$$< \varphi, +T \mid f> = {}_0\!< \varphi \mid e^{iH_{non\text{-}int} \cdot T} e^{-2iHT} e^{iH_{non\text{-}int} \cdot T} \mid \alpha>_0$$

$$\equiv {}_0\!< \varphi \mid S \mid \alpha>_0$$

where we defined the S-matrix

$$S = e^{iH_{non\text{-}int} T} e^{-2iHT} e^{iH_{non\text{-}int} \cdot T}$$

$$\left(= e^{iH_{non\text{-}int} \cdot t_{out}} \cdot e^{iH(t_{out} - t_{in})} \cdot e^{-iH_{non\text{-}int} \cdot t_{in}} \text{ generally} \right.$$

where in our case $t_{in} = -T$, $t_{out} = +T$). This holds when $T \to \infty$.

We drop the "0" subscripts from now on so that $|\alpha>$ and $|\varphi>$ are $t=0$ eigenstates of $H_{non\text{-}int}$. Physically, $S_{\alpha\varphi} = <\varphi| S |\alpha>$ then is the matrix element which describes the transition probability for scattering an initial state $|\alpha>$ to a state $|\varphi>$.

Note how S is unitary: $S^\dagger = S^{-1}$ so that $SS^\dagger = 1$. The S-matrix is usually written $S = 1 + iT$. The 1 corresponds to the "forward" part where there is no scattering*: the outgoing state is the same as the ingoing state evolved with $H_{non\text{-}int}$.

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* $S = 1$ is obtained for $H = H_{non\text{-}int}$.

$\mathcal{T}$ is called the T-matrix and contains information about scattering.

Due to the unitarity of S, we have

$$1 = (1 + i\mathcal{T})(1 - i\mathcal{T}^{\dagger}) \Rightarrow i\mathcal{T}\mathcal{T}^{\dagger} = \mathcal{T} - \mathcal{T}^{\dagger}.$$

In perturbation theory, one can neglect the l.h.s. since it is higher order in the interaction parameter λ than the r.h.s. Thus, to lowest order we have a Hermitian $\mathcal{T} = \mathcal{T}^{\dagger}$.

Note also that if the initial and final states are orthogonal, $\langle \psi | \varphi \rangle = 0$, we have: $\langle \psi | S | \varphi \rangle = i \langle \psi | T | \varphi \rangle$.

Finally, we will actually not assume normalized states $|\psi\rangle$ and $|\varphi\rangle$, as this will turn out to be inconvenient later. Leaving them unnormalized, we can compute the probability for a transition from $|\varphi\rangle$ to $|\psi\rangle$ as follows.

We have $|f\rangle = \sum_i c_i |\psi_{i,+T}\rangle$, but since $|\psi_{i,+T}\rangle$ are not normalized (but still orthogonal), $|c_i|^2$ is not the prob. that the system is in state i.

Instead:

$\langle f | f \rangle = \sum_i |c_i|^2 \langle \psi_{i,+T} | \psi_{i,+T} \rangle$. We can write this as

$1 = \sum_i P_i$ with $P_i = \dfrac{|c_i|^2 \langle \psi_{i,+T} | \psi_{i,+T} \rangle}{\langle f | f \rangle}$ and these can then be interpreted as proper probabilities.

Since $|f\rangle = e^{-2iHT}|\varphi\rangle$, we see that $\langle f|f\rangle = \langle\varphi|\varphi\rangle$.

The c_i-coeffs. themselves are found in the standard way:

$$c_i = \frac{\langle \psi_i, +T|f\rangle}{\langle \psi_i, +T|\psi_i, +T\rangle}$$

This means that the probability for the system to end up in a particular state $|\psi, +T\rangle$ when it started in $|\varphi\rangle$ at $t=-T$ is:

$$P_{\varphi\psi} = \frac{|\langle \psi, +T|f\rangle|^2}{\langle \psi, +T|\psi, +T\rangle\langle\varphi|\varphi\rangle} = \frac{|\langle\psi|S|\varphi\rangle|^2}{\langle\psi|\psi\rangle\langle\varphi|\varphi\rangle}$$

since $|\psi, +T\rangle = e^{-iH_{\text{non-int}} \cdot T}|\psi\rangle$. Thus, if $\langle\psi|\varphi\rangle = 0$ we get:

$$P_{\varphi\psi} = \frac{|\langle\psi|T|\varphi\rangle|^2}{\langle\psi|\psi\rangle\langle\varphi|\varphi\rangle}.$$

Without normalization, the correct completeness relation is also

$$\sum_i \frac{|\psi_i\rangle\langle\psi_i|}{\langle\psi_i|\psi_i\rangle} = 1.$$

The LSZ reduction formula

We now demonstrate how to relate the S-matrix between two widely separated states to the correlators we have spent so much time studying.

To be concrete, we consider our φ^4-theory with a real scalar field.
Let the incoming and outgoing states be

$|\varphi\rangle = |\vec{k}_1, \vec{k}_2, \dots \vec{k}_n\rangle$ and $|\psi\rangle = |\vec{p}_1, \vec{p}_2, \cdots \vec{p}_m\rangle$.

Symmetry is always a powerful argument and from the translational symmetry of our theory at asymptotic times (no interaction between the particles) we can immediately state that for a one-particle dressed state, we must have (up to a phase)

$\langle 0| \varphi_R(x) |\vec{k}\rangle = e^{-ikx}$ where $\varphi_R(x)$ is the renormalized scalar field. The e^{-ikx} factor results from translational invariance for the following reason.

P^μ (momentum operator) is the generator of translations since

$$e^{iaP} \varphi_R(x) = \varphi_R(x+a) e^{ia\cdot P} \Rightarrow$$

$$\langle 0|\varphi_R(x+a)|\vec{k}\rangle = \langle 0| e^{iaP} \varphi_R(x) e^{-iaP} |\vec{k}\rangle = \langle 0| 1\cdot \varphi_R(x)\, e^{-iak} |\vec{k}\rangle$$
$$= \langle 0|\varphi_R(x)|\vec{k}\rangle e^{-iak},$$

which has the solution $\langle 0|\varphi_R(x)|\vec{k}\rangle = e^{-ikx}$.

You may also be consolidated by the fact that in the free case $(\lambda=0)$, we have the same relation: in the free case we had

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega(\vec{k})}} \left(a_{\vec{k}} e^{-ikx} + a^\dagger_{\vec{k}} e^{ikx} \right) \text{ and}$$

$|\vec{k}\rangle = \sqrt{2\omega(\vec{k})}\, a^\dagger_{\vec{k}} |0\rangle$ with $[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = (2\pi)^3 \delta(\vec{k}-\vec{k}')$. Insert and get:

$$\langle 0 | \varphi(x) | \vec{k} \rangle_{free} = e^{-ikx}.$$

For free particles, we can actually extend this relation to:

$$\langle \vec{p}_1, \vec{p}_2, \ldots \vec{p}_m | : \varphi(x_1) \ldots \varphi(x_n) \varphi(x_{n+1}) \ldots \varphi(x_{n+m}) : | \vec{k}_1, \vec{k}_2, \ldots \vec{k}_n \rangle_{free}$$

$$= \sum_{\sigma \in S_{n+m}} \exp\left(-i \sum_{j=1}^{n+m} k_j \cdot x_{\sigma(j)} \right)$$

where $: \,:$ means normal-ordering ($a^\dagger$ to the left, a to the right).

For instance, $: a^\dagger a a a^\dagger a : = a^\dagger a^\dagger a a a$. Moreover, σ is an element of the permutation group on $n+m$ objects and $\sigma(j)$ is the j'th element in σ.

We defined $k_{n+j} \equiv -p_j$. To see why the $\sum_{\sigma \in S_{n+m}}$ appears, consider the case $n=2, m=2$ and recall that S_{n+m} has $(n+m)!$ ($=24$ in this case) elements. Considering only the operator structure of $:\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4):$ and discarding all terms that do not give a finite result, we have:

$$:(a_1+a_1^\dagger)(a_2+a_2^\dagger)(a_3+a_3^\dagger)(a_4+a_4^\dagger): = :(a_1a_2 + a_1a_2^\dagger + a_1^\dagger a_2 + a_1^\dagger a_2^\dagger)(a_3a_4 + a_3a_4^\dagger + a_3^\dagger a_4 + a_3^\dagger a_4^\dagger):$$

$$= :(a_1a_2a_3^\dagger a_4^\dagger + a_1a_2^\dagger a_3 a_4^\dagger + a_1a_2^\dagger a_3^\dagger a_4 + a_1^\dagger a_2 a_3 a_4^\dagger + a_1^\dagger a_2 a_3^\dagger a_4 + a_1^\dagger a_2^\dagger a_3 a_4):$$

$$= a_3^\dagger a_4^\dagger a_1 a_2 + a_2^\dagger a_4^\dagger a_1 a_3 + a_2^\dagger a_3^\dagger a_1 a_4 + a_1^\dagger a_4^\dagger a_2 a_3 + a_1^\dagger a_3^\dagger a_2 a_4 + a_1^\dagger a_2^\dagger a_3 a_4.$$

These are 6 terms. For each term, $a^\dagger a^\dagger$ should create $\vec{p}_1$ and $\vec{p}_2$ which can be done in two ways, while aa should annihilate $\vec{k}_1$ and $\vec{k}_2$ which can also be done in two ways. Combined, $6 \times 4 = 24 = 4!$ terms.

If $k_i \neq -k_j$ for all i and j, we can drop the normal ordering.
This criterion basically expresses that none of the $\vec{k}$'s should be equal to any of the $\vec{p}$'s, otherwise the normal-ordering operation will make a difference. To see this, consider the simple case:

$<\vec{p}_1 | \varphi(x_1)\varphi(x_2) | \vec{k}_1>$ and focus only on the operator content:

$$= <\vec{p}_1 | \iint d^3k\, d^3k' \, (a_{\vec{k}} + a_{\vec{k}}^{+})(a_{\vec{k}'} + a_{\vec{k}'}^{+}) | \vec{k}_1 > .$$

The only way to get a non-zero result is from $a_{\vec{k}} \cdot a_{\vec{k}'}^{+}$ with $\vec{k}' = \vec{p}_1$ and $\vec{k} = \vec{k}_1$ & $a_{\vec{k}}^{+} a_{\vec{k}'}$ with $\vec{k} = \vec{p}_1$ and $\vec{k}' = \vec{k}_1$.
(use that $|\vec{k}_1> = \sqrt{2\omega(\vec{k}_1)}\, a_{\vec{k}_1}^{+} |0>$). The $a^{+}a$ term is already normal-ordered, and aa^{+} is only equal to $:aa^{+}: = a^{+}a$ if $\vec{k}_1 \neq \vec{p}_1$ so that the operators commute. This holds if $k_i \neq -k_j$ (with our definition of $k_{n+j} \equiv -p_j$).

It is also useful to note the relation

$$e^{-ikx} = \int d^4q \, \delta(q-k)\, e^{-iqx} \frac{i}{q^2 - m^2 + i\varepsilon} \frac{k^2 - m^2 + i\varepsilon}{i}$$

$$= \iint d^4y \frac{d^4q}{(2\pi)^4} e^{-iq(x-y)} \frac{i}{q^2 - m^2 + i\varepsilon} \frac{k^2 - m^2 + i\varepsilon}{i} \cdot e^{-iky}$$

$$= \int d^4y \, G_F(x-y) \frac{k^2 - m^2 + i\varepsilon}{i} e^{-iky}$$

To proceed with establishing a connection between S-matrix elements (transition probabilities) and correlators, we are going to need the general version of <u>Wick's theorem</u>.

We first define what is meant mathematically by a <u>contraction</u>.

For two ladder operators A and B, their contraction $\overset{\sqcap}{AB}$ (also denoted $\overline{AB}$, $\dot{A}\dot{B}$) is defined as:

$$\overset{\sqcap}{AB} \equiv AB - :AB: \quad \text{where } : \text{ denotes normal-ordering.}$$

One also uses the notation $:AB: = N\{AB\}$. Recall that normal-ordering places all creation operators to the left of annihilation operators with as few shifts in position of operators as possible (thus the internal order of all $a^\dagger$ and a does not change, respectively). Using bosonic $a, a^\dagger$ as an example, we have $[a_i, a_j^\dagger] = \delta_{ij}$ and thus:

$$\overset{\sqcap}{a_i a_j} = a_i a_j - :a_i a_j: = 0$$

$$\overset{\sqcap}{a_i a_j^\dagger} = a_i a_j^\dagger - :a_i a_j^\dagger: = a_i a_j^\dagger - a_j^\dagger a_i = \delta_{ij}.$$

Note that for fermion operators, an extra (-1) multiplicative factor appears for each time two operators change place. But the definition of $\sqcap$ given above remains true and we'll focus for now only on the bosonic case.

Products of operators now turn out to obey a systematic expansion using normal-ordering and contractions. We will see how this is useful when evaluating GS expectation values (such as for correlators) since the exp. value of a normal-ordered product of operators in the GS is clearly zero: $\langle 0 | :ABCD\ldots: | 0 \rangle = 0$.

To observe this system, note that:

$$a_i a_j^\dagger a_k = a_j^\dagger a_i a_k + \delta_{ij} a_k = :a_i a_j^\dagger a_k: + \overbracket{a_i a_j^\dagger} a_k = :a_i a_j^\dagger a_k: + :\overbracket{a_i a_j^\dagger} a_k:$$
$$\underbrace{+ :\overbracket{a_i a_j^\dagger a_k}: + :a_i \overbracket{a_j^\dagger a_k}:}_{=0}$$

added by hand since it is zero

and

$$a_i a_j^\dagger a_k a_l^\dagger = a_j^\dagger a_i a_l^\dagger a_k + \delta_{kl} a_j^\dagger a_i + \delta_{ji} a_l^\dagger a_k + \delta_{ij} \delta_{kl}$$
$$= \text{(moving terms so that we get normal-ordered 1st term)}$$
$$= :a_i a_j^\dagger a_k a_l^\dagger: + :\overbracket{a_i a_j^\dagger a_k a_l^\dagger}: + :a_i a_j^\dagger \overbracket{a_k a_l^\dagger}:$$
$$+ :\overbracket{a_i a_j^\dagger} a_k a_l^\dagger: + :\overbracket{a_i a_j^\dagger}\,\overbracket{a_k a_l^\dagger}:$$
$$+ \underbrace{:\overbracket{a_i a_j^\dagger a_k}\!\!\overbracket{ a_l^\dagger}: + :\overbracket{a_i a_j^\dagger a_k} a_l^\dagger: + :a_i \overbracket{a_j^\dagger a_k a_l^\dagger}:}_{=0}$$

← added by hand

The general statement of the theorem is then (valid both for bos. and ferm.):

$$\boxed{ABCD\ldots = :ABCD\ldots: + \sum_{\substack{\text{single}\\ \text{contractions;}\\ \text{all distinct pairs}}} :\overbracket{AB}CD\ldots: + \sum_{\substack{\text{double}\\ \text{contractions;}\\ \text{distinct}\\ \text{pairs}}} :\overbracket{AB}\,\overbracket{CD}\ldots: + \ldots}$$

Let's now apply Wick's theorem to fields $\varphi(x)$ and see how that works. This will be very useful when evaluating various types of correlators. To start with, recall that

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega(\vec{k})}} \left(a_{\vec{k}} e^{-ikx} + a^+_{\vec{k}} e^{ikx} \right)$$

for our real, scalar field. We write this now as:

$$\varphi(x) = \varphi^+(x) + \varphi^-(x) \text{ with } \varphi^+(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{k}}}} a_{\vec{k}} e^{-ikx}$$

and $\varphi^-(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{k}}}} a^+_{\vec{k}} e^{ikx}$. We then have that for $x^0 > y^0$:

$$T\{\varphi(x)\varphi(y)\} = \varphi(x)\varphi(y) = [\varphi^+ + \varphi^-][\varphi^+ + \varphi^-]$$

$$= \varphi^+(x)\varphi^+(y) + \varphi^-(x)\varphi^+(y) + \varphi^-(y)\varphi^+(x) + [\varphi^+(x), \varphi^-(y)] + \varphi^-(x)\varphi^-(y)$$

Here, we managed to normal-order the last line at the expense of picking up a c-term $[\varphi^+(x), \varphi^-(y)]$. The fact that this is a c-number can be seen by using $[a_{\vec{k}}, a^+_{\vec{k}'}] = (2\pi)^3 \delta(\vec{k} - \vec{k}')$:

$$[\varphi^+(x), \varphi^-(y)] = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} e^{-ik(x-y)} \equiv D(x-y).$$

But as we saw in the first chapter, this is also the result of evaluating $\langle 0|\varphi(x)\varphi(y)|0\rangle$:

$$\langle 0|\varphi(x)\varphi(y)|0\rangle = D(x-y).$$

Thus, we have shewn that for $x^0 > y^0$:

$$T\{\varphi(x)\varphi(y)\} = :\varphi(x)\varphi(y): + D(x-y).$$

Repeat the calculation for $y^0 > x^0$:

$$T\{\varphi(x)\varphi(y)\} = :\varphi(x)\varphi(y): + D(y-x).$$

But the full Feynman propagator is given by precisely

$$G_F(x-y) = \Theta(x^0-y^0)\,D(x-y) + \Theta(y^0-x^0)\,D(y-x)$$

as we shewed in chapter 1 $[\,G_F(x-y) = \langle 0|\,T\{\varphi(x)\varphi(y)\}\,|0\rangle\,]$, so in general we have:

$$T\{\varphi(x)\varphi(y)\} = :\varphi(x)\varphi(y): + G_F(x-y).$$

In analogy with how we defined a contraction for the $a, a^\dagger$ operators, we now define a contraction of a pair of fields φ as:

$$\Leftarrow \quad \overline{\varphi(x)\varphi(y)} \equiv T\{\varphi(x)\varphi(y)\} - :\varphi(x)\varphi(y): \qquad (= G_F(x-y)).$$

Note how the contraction equals $G_F(x-y)$ even without taking expectation value in the GS $|0\rangle$: the above is an operator relation. *

Generalization

We can now generalize this to any number of fields:

$$\boxed{T\{\varphi_1\varphi_2\ldots\varphi_n\} = :\varphi_1\varphi_2\ldots\varphi_n: + :\text{all possible contractions}:}$$

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* Similar procedure for complex scalar field: $T\{\varphi(x)\varphi^\dagger(y)\} = :\varphi(x)\varphi^\dagger(y): + \Delta_F(x-y)$
with $\overline{\varphi(x)\varphi^\dagger(y)} \equiv \Delta_F(x-y)$ and $\overline{\varphi(x)\varphi(y)} = \overline{\varphi^\dagger(x)\varphi^\dagger(y)} = 0$.

For instance, for $n=4$ we'd get:

$$T\{\varphi_1\varphi_2\varphi_3\varphi_4\} = :\varphi_1\varphi_2\varphi_3\varphi_4: + \overbracket{\varphi_1\varphi_2}\,:\varphi_3\varphi_4: + \overbracket{\varphi_1\varphi_3}\,:\varphi_2\varphi_4:$$

$$+ \text{ 4 similar terms}$$

$$+ \overbracket{\varphi_1\varphi_2}\,\overbracket{\varphi_3\varphi_4} + \overbracket{\varphi_1\varphi_3}\,\overbracket{\varphi_2\varphi_4} + \overbracket{\varphi_1\varphi_4}\,\overbracket{\varphi_2\varphi_3}$$

The proof of the generalization of Wick's theorem goes by induction. We have seen that it's true for $n=2$. Assume it's true for $n-1$ fields $\varphi_2 \cdots \varphi_n$. If we can show that it then is true also for n fields $\varphi_1\varphi_2 \cdots \varphi_n$, we have proven it in general.

So we add φ_1 to $\varphi_2 \cdots \varphi_n$ and take $x_1^0 > x_k^0$ for all $k=2,\dots n$. No loss in generality since we can move boson operators around inside $T\{\dots\}$ as much as we like and still get the same result after time-ordering. This allows us to pull φ_1 out to the left of the time-ordering:

$$T\{\varphi_1\varphi_2 \dots \varphi_n\} = (\varphi_1^+ + \varphi_1^-)\,T\{\varphi_2, \dots \varphi_n\}$$

$$= (\varphi_1^+ + \varphi_1^-)\,[\,:\varphi_2 \cdots \varphi_n: + :\text{contractions}:\,] \quad (*)$$

(since the theorem is true for $n-1$ fields)

Now, we need to get φ_1 inside the normal ordering and partaking in the contractions somehow.

The φ_1^- term is easy: we can just move it into both terms

since $\varphi_1^- : ABCD\ldots: \;=\; :\varphi_1^- ABCD\ldots:$ (recall $\varphi_1^- = \int \frac{d^3k}{(2\pi)^3} a_k^+ e^{ikx} \frac{1}{\sqrt{2\omega_k}}$).

For the φ_1^+ term, it has to be commuted to the right past all other φ's in order to earn a place inside the normal-ordering.

Consider first the term with no contractions.

$$\varphi_1^+ :\varphi_2\ldots\varphi_n: \;=\; :\varphi_2\ldots\varphi_n:\varphi_1^+ + [\varphi_1^+, :\varphi_2\ldots\varphi_n:]$$

$$= :\varphi_1^+\varphi_2\ldots\varphi_n: + \underbrace{:[\varphi_1^+,\varphi_2]\varphi_3\ldots\varphi_n + \varphi_2[\varphi_1^+,\varphi_3]\varphi_4\ldots\varphi_n + \ldots:}$$

This follows by $\overset{\text{proof via}}{\text{induction}}$ on this particular term and by using $[A,BC] = [A,B]C + B[A,C]$.

Now in each commutator $[\varphi_1^+,\varphi_j] = [\varphi_1^+,\varphi_j^-]$ since $[\varphi_1^+,\varphi_j^+]=0$.

But since $x_1^0 > x_j^0$ for all $j=2,\ldots n$, we have:

$$[\varphi_1^+,\varphi_j] = \overline{\varphi_1\varphi_j} \qquad (\text{since } \overline{\varphi_1\varphi_j} = [\varphi_1^+,\varphi_j^-] \text{ for } x_1^0 > x_j^0)$$

So we have found that

$$\varphi_1^+ :\varphi_2\ldots\varphi_n: \;=\; :\varphi_1^+\varphi_2\ldots\varphi_n + \overline{\varphi_1\varphi_2}\ldots\varphi_n + \overline{\varphi_1\varphi_2\varphi_3}\ldots\varphi_n + \ldots:$$

The first term on the rhs combines with $:\varphi_1^-\varphi_2\ldots\varphi_n:$ to give $:\varphi_1\varphi_2\ldots\varphi_n:$. That's then the first term on the rhs. of Wick's theorem for $T\{\varphi_1\varphi_2\ldots\varphi_n\}$. The rest of the terms on the rhs above give all possible terms involving a single contraction of φ_1 with another field.

Next, we consider φ_1^+ : all contractions not involving φ_1 : , which is the remaining term in (x). Using exactly the same procedure as we just showed for the terms in φ_1^+: all contr. not involving φ_1 : that involve one contr., this will produce all possible terms including that contraction and that contraction + a contraction of φ_1 with one of the other fields.

Doing this with all remaining terms eventually gives us all possible contractions, including the φ_1 term, which concludes the proof.

Free vs. interacting theory: higher-order correlators

A key result regarding using Wick's theorem for higher-order correlators is that

- These result in products of two-point correlators for a free theory : $\langle 0 | T\{\varphi_1 \varphi_2 \cdots \varphi_n\} | 0 \rangle_{\text{free}} =$ products of 2 point.

  The reason is that the normal-ordered products give zero when acting on $|0\rangle_{\text{free}}$.

- The same is not true for an interacting-theory higher-order corr. $\langle 0 | T\{\varphi_1 \cdots \varphi_n\} | 0 \rangle$ because the normal-ordered product does not necessarily vanish when acting on the GS $|0\rangle$ of the interacting theory (which is different from $|0\rangle_{\text{free}}$).

Now back to the S-matrix discussion.

Using Wick's theorem and the previously mentioned relation for e^{-ikx}, it follows that if $k_i \neq -k_j$ for all i and j, then:

$$\sum_{\sigma \in S_{n+m}} \exp\left(-i \sum_{j=1}^{n+m} k_j \cdot x_{\sigma(j)}\right)$$

$$= \int \prod_{j=1}^{n+m} \left(d^4 y_j \, \frac{k_j^2 - m^2 + i\varepsilon}{i} \, e^{-i k_j \cdot y_j} \right) \langle T\{ \varphi(y_1) \varphi(y_2) \ldots \varphi(y_{n+m})$$

$$\times \quad \varphi(x_1) \ldots \varphi(x_{n+m}) \} \rangle_{free}$$

See exercise (proof below).

PROOF FOR EXERCISE (Case $n=1, m=1$).

$$\int \prod_{j=1}^{2} d^4 y_j \, \frac{k_j^2 - m^2 + i\varepsilon}{i} \, e^{-i k_j y_j} \quad \langle T\{ \varphi(y_1) \varphi(y_2) \varphi(x_1) \varphi(x_2) \} \rangle_{free}$$

$$= \; - '' - \; \Big[\underbrace{\langle : \varphi(y_1) \varphi(y_2) \varphi(x_1) \varphi(x_2) : \rangle_{free}}_{=0 \text{ since } \langle 0|ABC\ldots:|0\rangle = 0} + \underbrace{\text{terms with two fields contracted}}_{=0 \text{ for same reason}}$$

$$+ \; G_F(y_1 - y_2) \, G_F(x_1 - x_2) + G_F(y_1 - x_1) \, G_F(y_2 - x_2) + G_F(y_1 - x_2) \, G_F(y_2 - x_1) \Big]$$

Looking back at our definition of $G_F(x-y)$, we see that it satisfies: $G_F(x-y) = G_F(y-x)$. Therefore, we have:

$$= \left[\int \prod_{j=1}^{2} d^4 y_j \, \frac{k_j^2 - m^2 + i\varepsilon}{i} \, e^{-i k_j y_j} \, G_F(y_1 - y_2) \, G_F(x_1 - x_2) \right] + e^{-i(k_1 x_1 + k_2 x_2)} + e^{-i(k_1 x_2 + k_2 x_1)}$$

We solve the first integral by shifting measure from $d^4 y_1 \, d^4 y_2$ to $d^4 y \, d^4 y_2$ with $y = y_1 - y_2$ being the relative coordinate.

The Jacobian of this transformation is

$$\det\begin{pmatrix} \frac{\partial y}{\partial y_1} & \frac{\partial y}{\partial y_2} \\ \frac{\partial y_2}{\partial y_1} & \frac{\partial y_2}{\partial y_2} \end{pmatrix} = \det\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = 1 \text{, so } d^4y\, d^4y_2 = d^4y_1\, d^4y_2 .$$

We obtain that.

$$\iint d^4y\, d^4y_2 \; \frac{k_1^2 - m^2 + i\varepsilon}{i} \; \frac{k_2^2 - m^2 + i\varepsilon}{i} \; G_F(y)\, e^{-ik_1y_1 - ik_2y_2} \cdot G_F(x_1 - x_2)$$

$$= \;\text{-''-}\; G_F(y)\, e^{-ik_1(y_1 - y_2) - iy_2(k_1 + k_2)} \cdot G_F(x_1 - x_2).$$

$$= \left[\int d^4y_2 \; \tilde{G}_F(-k_1) \cdot e^{-iy_2(k_1 + k_2)}\right] G_F(x_1 - x_2)$$

$$= (2\pi)^4 \delta(k_1 + k_2)\, G_F(x_1 - x_2)\, \tilde{G}_F(-k_1).$$

This is zero since we assumed that $k_1 \neq -k_2$. $\square$

We will find use for this relation later. Next, we note that we previously found that

$$\langle 0 | e^{-2iHT} | 0 \rangle = Z(0) = \langle 0 | T\{ e^{i\int d^4x\, \mathcal{L}_I} \} | 0 \rangle_{free} \cdot Z_{free}(0)$$

under the assumption that $H_0 |0\rangle = 0^*$ and $H = H_0 + H_I$. We derived this using a path integral argument and $|0\rangle$ is the GS of the interacting theory whereas $|0\rangle_{free}$ is the GS for the free theory.

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* We used that $H_0 = \int \frac{d^3k}{(2\pi)^3} \omega(\vec{k}) a_{\vec{k}}^\dagger a_{\vec{k}} + C$ and ignored C.

Let us now generalize this to other states than vacuum via a different argument.

In QM, we know that there are two commonly used "pictures" to describe states and operators.

Schr. pic.: all time-dep. in states while operators have no time dep. (unless the time-dep. is explicit, like for time-evolution op. $\hat{U}$) Inner products have the form $\langle \psi, t | \hat{O} | \psi, t \rangle$.

Heisenberg pic.: no time-dep. in states, all time-dep. in operators. Inner products have the form $\langle \psi | \hat{O}_H(t) | \varphi \rangle$.

Pictures equivalent w.r.t. phys. observables since

$$\langle \psi | \hat{O}_H(t) | \varphi \rangle = \langle \psi | e^{iHt} \hat{O} e^{-iHt} | \varphi \rangle = \langle \psi, t | \hat{O} | \varphi, t \rangle .$$

But a third option is to use the interaction picture where we put part of the time-dep. in states and part in ops. In the interaction picture, the time-dep. of an operator is governed by H_0:

$\hat{O}(t) \equiv e^{iH_0 t} \hat{O} e^{-iH_0 t}$ while H_I gives the time-dep. of the states.

Apply this now to the op. $e^{iH_0 T} e^{-2iHT} e^{iH_0 T}$. First re-express as:

$$e^{iH_0 T} e^{-2iHT} e^{iH_0 T} = \lim_{\Delta t \to 0} e^{iH_0 T} \left(e^{-iH_0 \Delta t} e^{-iH_I \Delta t} \right)^{\frac{2T}{\Delta t}} e^{iH_0 T}$$

with $H = H_0 + H_I$. Then use: $e^{-iH_0\Delta t} = e^{-iH_0(t+\Delta t)} e^{iH_0 t}$ where the t is determined by the position of the particular $e^{-iH_0\Delta t}$ in the product.

We then have:

$$e^{iH_0T} e^{-2iHT} e^{iH_0T} = \lim_{\Delta t \to 0} e^{iH_0T} \left[e^{-iH_0(t_1+\Delta t)} e^{iH_0 t_1} e^{-iH_I\Delta t} \right]$$

$$\times \left[e^{-iH_0(t_2+\Delta t)} e^{iH_0 t_2} e^{-iH_I\Delta t} \right] \times \cdots \left[e^{-iH_0(t_n+\Delta t)} e^{iH_0 t_n} e^{-iH_I\Delta t} \right] e^{iH_0T}$$

Here, we choose t_i so that $t_1 = T,\ t_2 = T - \Delta t, \ldots\ t_n = -T$. (*)

Since the first three factors $e^{iH_0T} \cdot e^{-iH_0(t_1+\Delta t)} \to 1$ in the limit $\Delta t \to 0$,

$$e^{iH_0T} \cdot e^{-2iHT} \cdot e^{iH_0T} = \lim_{\Delta t \to 0} \prod_t \left(e^{iH_0 t} e^{-iH_I\Delta t} e^{-iH_0 t} \right)$$

$$= T\left\{ \lim_{\Delta t \to 0} \prod_t e^{-iH_I(t)\Delta t} \right\}$$ since the t's are time-ordered from the outset.

$$= T\left\{ e^{-i\int dt\, H_I(t)} \right\}$$

$$= T\left\{ e^{i\int d^4x\, \mathcal{L}_I} \right\}$$ since $L_I = \int d^3x\, \mathcal{L}_I = -H_I$. (†)

Therefore, we have proven that for any states, we have:

$$\langle\psi| e^{iH_0T} e^{-2iHT} e^{iH_0T} |\varphi\rangle = \langle\psi| T\left\{ e^{i\int d^4x\, \mathcal{L}_I} \right\} |\varphi\rangle$$

where the time-dependence of $\mathcal{L}_I$ is then determined by H_0 (just like we're done all the way up to now since we use the same field expansion of $\varphi(x)$ in terms of e^{-ikx} etc. in $\mathcal{L}_I$ despite the presence of interactions. So it's the same $\mathcal{L}_I(x)$ that we're used to)

(†) $L_I = -H_I$ is true when L_I does not depend on derivatives of fields and when L_0 is quadratic in the fields:
$\mathcal{L}[\varphi, \partial\varphi] = \frac{1}{2}(\partial\varphi)^2 - \mathcal{L}_{int}[\varphi] \Rightarrow \mathcal{H} = \partial\varphi \frac{\delta\mathcal{L}}{\delta(\partial\varphi)} - \mathcal{L}[\varphi, \partial\varphi] = \frac{1}{2}(\partial\varphi)^2 + \mathcal{L}_{int}[\varphi]$.

(*) This is allowed since the choice of t's is arbitrary, so we choose them to be time-ordered.

We now come back to a statement we made initially, namely that there exists a non-interacting Hamiltonian with "dressed" particles that have the physically correct mass and residue. Our claim is now that this Hamiltonian should actually be the free Hamiltonian H_0!

Let us see why.

What we did in the 3+1 dim. theory was to introduce counterterms $\delta_Z, \delta_{m^2}, \delta_\lambda$ in $\mathcal{L}$ in order to fix the physical mass at m and the residue of the propagator at 1. After adding these counterterms, the Lagrangian would take the form:

$$\mathcal{L}(x) = \frac{1}{2}(1+\delta_Z)\,\partial_\mu\varphi\,\partial^\mu\varphi - [(1+\delta_Z)m^2+\delta_{m^2}]\varphi^2 - \frac{1}{4!}(\lambda+\delta\lambda)\varphi^4$$

Here, λ is not the bare interaction: $\lambda+\delta\lambda$ is, as we commented earlier. Similarly, $\varphi(x)$ is not the bare field, but $\varphi\cdot Z^{1/2}$ with $Z=1+\delta_Z$ is. Instead, $\varphi(x)$ is the renormalized field: the counterterms were chosen so that the pole and residue of the propagator $\int d^4x\, e^{-ikx}\langle T\{\varphi(x)\varphi(0)\}\rangle$ stayed at m^2 and 1, so $\varphi(x)$ has to be the renormalized field.

So how do we get a propagator with the same physical properties, but without any interactions in the theory? The answer is simple: we turn off λ and remove all counterterms (which have to vanish when $\lambda=0$).

This gives us a free Lagrangian $\mathcal{L}$ corresponding exactly to the free Hamiltonian H_0. In other words, removing the counterterms and setting $\lambda=0$ in fact is the way to make the separated, non-interacting particles still be "dressed" (interact with virtual particles surrounding them) since our propagator has the correct physical mass and residue in this case. If we hadn't removed the counterterms δ_Z and δ_{m^2} in the free part of $\mathcal{L}$, particles would be created by the bare field $Z^{1/2}\cdot\varphi(x)$ and they would have mass $(1+\delta_Z)m^2+\delta m^2$.

So it seems we should let $H_{\text{non-int}} \to H_0$ and then be done. However, one detail remains. The S-matrix is actually only defined up to a phase-factor $e^{i\alpha}$ which does not destroy its unitarity ($e^{i\alpha^\dagger}=e^{-i\alpha}$) and which does not change the physics (observables such as scatt. cross-section) because observables depend on $|S_{\alpha\beta}|^2=|\langle\psi|S|\varphi\rangle|^2$.

This phase-factor is fixed by demanding that the S-matrix leaves the vacuum-state for H_0 (free case) invariant: $\langle 0|S|0\rangle_{\text{free}}=1$. This is reasonable if vacuum is a unique state that is separated from excited states by an energy gap (thus energy conservation should preserve it under a time-evolution which S basically is).

So we have $S=e^{i\alpha}\cdot e^{iH_0T}e^{-2iHT}e^{iH_0T}$ and demand $\langle 0|S|0\rangle_{\text{free}}=1$.

This gives $e^{i\alpha} = \left(\langle 0 | T\{ e^{i\int d^4x\, \mathcal{L}_I(x)} \} | 0 \rangle_{\text{free}} \right)^{-1}$.

Now, we can write down the S-matrix element we've been studying:

$$\langle \psi | S | \varphi \rangle = \langle \vec{p}_1, \dots \vec{p}_m | S | \vec{k}_1, \dots \vec{k}_n \rangle_{\text{free}}$$

since $|\varphi\rangle$ and $|\psi\rangle$ were eigenstates of $H_{\text{non-int}}$ which we replaced with H_0

$$= \frac{\langle \vec{p}_1 \cdots \vec{p}_m | T\{ \exp\left(i\int d^4x\, \mathcal{L}_I(x)\right) \} | \vec{k}_1, \dots \vec{k}_n \rangle_{\text{free}}}{\langle 0 | T\{ \exp\left(i\int d^4x\, \mathcal{L}_I(x)\right) \} | 0 \rangle_{\text{free}}}$$

Now we're getting closer to being able to express the S-matrix element in terms of correlators! To proceed, we assume with very little loss of generality $\vec{p}_i \neq \vec{k}_j$: all particles have at least some deflection in the scattering. It follows that

(*)

$$\langle \vec{p}_1 \cdots \vec{p}_m | : \varphi(x_1) \cdots \varphi(x_N) : | \vec{k}_1, \dots \vec{k}_n \rangle_{\text{free}} = 0$$

unless $N = m + n$, since otherwise there aren't enough fields to e.g. remove all $\{\vec{p}\}$ states and remove all $\{\vec{k}\}$-states so that we get vacuum in both bra and ket and thus a finite overlap in the expectation value.

(*) When $n \neq m$, we should understand that a finite overlap is obtained when vacuum is obtained in both bra and ket (or equivalently that one first destroys the $\{\vec{k}\}$-states in ket and then create all $\{\vec{p}\}$-states in the ket.)

What we should then do is to use Wick's theorem on $T\{e^{i\int d^4x \mathcal{L}_I(x)}\}$

$= :e^{i\int d^4x \mathcal{L}_I(x)}: +$ all possible contractions and only keep the terms

that have n+m ordered fields. Let us denote the contribution from

these terms $:F(\{\varphi\}):$ where F is then a sum over terms that all

have n+m fields φ. So we have

$$\langle \vec{p}_1, \dots \vec{p}_m | S | \vec{k}_1, \dots \vec{k}_n \rangle_{free}$$

$$= \frac{\langle \vec{p}_1, \dots \vec{p}_m | :F(\{\varphi\}): | \vec{k}_1, \dots \vec{k}_n \rangle_{free}}{\langle 0 | T\{e^{i\int d^4x \mathcal{L}_I(x)}\} | 0 \rangle_{free}}.$$

But we showed previously that if $\vec{p}_i \neq \vec{k}_j$ for all i and j, then the normal-ordering does not matter

and we can express the nominator as:

$$\langle \vec{p}_1, \dots | :F(\{\varphi\}): | \vec{k}_1, \dots \rangle_{free} = \langle \vec{p}_1 \dots | F(\{\varphi\}) | \vec{k}_1, \dots \rangle_{free}$$

$$= \int \prod_{j=1}^{n+m} d^4 y_j \frac{k_j^2 - m^2 + i\epsilon}{i} e^{-ik_j y_j} \langle 0 | T\{\varphi(y_1) \dots \varphi(y_{n+m}) F(\{\varphi\})\} | 0 \rangle_{free} \quad (\star)$$

This finally establishes the connection between S-matrix elements and the correlators we have been spending so much time focusing on!

The expression above is thus equal to $\langle \psi | S | \varphi \rangle$ and we found earlier that the transition probability was $P_{\varphi\psi} = \frac{|\langle \psi | S | \varphi \rangle|^2}{\langle \psi | \psi \rangle \langle \varphi | \varphi \rangle}$.

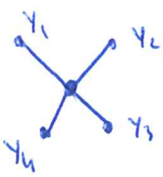
The S-matrix element can now be linked directly to an experimentally measurable quantity: the scattering cross section σ which depends on P_{exp} (prop. to). You have dealt extensively with these in particle physics, so we won't do the same here, but now you are actually able to <u>derive</u> the mathematical expression for the Feynman amplitude associated with different scattering diagrams (instead of just learning rules for how this amplitude is obtained from a diagram, as in part. phys.).

Our derivation of the S-matrix element is thus where the math. rules for how to get amplitudes come from!

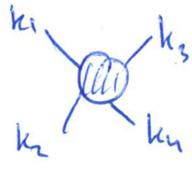
In (★), we will have no contribution from contractions of $\varphi(y_i)$ and $\varphi(y_j)$, as we showed before (see exer.). Let's show one example of a diagram that contributes for the case $n=2, m=2$:

$$\sim \langle 0 | T\{\varphi_1 \dots \varphi_4 \; i\int d^4x \, :\lambda \cdot \varphi^4(x):\}|0\rangle_{free} = i\lambda \int d^4x \, \langle 0 | G(y_1-x)\, G(y_2-x)\, G(y_3-x)\, G(y_4-x) |0\rangle_{free}$$

$$+ \dots = i\lambda \int d^4x \, G(y_1-x) \cdots G(y_4-x) + \dots$$ which according to our Feynman rules is



. After also taking into account the $\int d^4y_j$ integration and $\frac{k_j^2-m^2+i\varepsilon}{i}$ factors, it turns out that we end up with connected, truncated 4-point ($n+m=2+2=4$) correlators of the type



, but we won't go into the details here: the connection between physically observable S-matrix elements (via cross section σ) and correlators has now been established.