3+1 dimensional Dirac fermions

The setup is almost the same as in 0+1 dimensions; but we now have to endow the fermion fields with a Dirac index. The meaning of the functional differentials thus has to be:

$$\mathcal{D}\bar{\psi}\,\mathcal{D}\psi = \prod_x \prod_{\alpha=1}^{4} d\bar{\psi}_\alpha(x)\, d\psi_\alpha(x)$$

where $\bar{\psi} = \psi^\dagger \gamma^0$ means that $\bar{\psi}_\alpha(x) = [\psi^\dagger(x)]_\beta\, \gamma^0_{\beta\alpha}$ in component-form.

The functional integral including the source terms now has the form:

$$Z(\eta,\bar{\eta}) = \int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi \, \exp\left[i \int d^4x \left[\bar{\psi}(x)(i\partial\!\!\!/ - m + i\varepsilon)\psi(x) + \bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x) \right] \right]$$

where the sources also have Dirac indices. As we proved a bit earlier, we can complete the square and shift integration variables for Grassmann numbers. This gives

$$Z(\eta,\bar{\eta}) = \int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi \, \exp\left(i \int d^4x \, [\bar{\psi}(x) - i\bar{\eta} S_F(x)](i\partial\!\!\!/ - m + i\varepsilon)[\psi(x) - iS_F \eta(x)] \right)$$

$$\times \exp\left(- \int d^4x \, \bar{\eta}(x) S_F \eta(x) \right)$$

$$= Z(0,0) \cdot \exp\left(- \int d^4x \int d^4y \, \bar{\eta}(x) S_F(x-y) \eta(y) \right)$$

where we computed $S_F(x-y)$ previously: (it is a matrix)

$$[S_F(x-y)]_{\alpha\beta} = \int \frac{d^3k}{(2\pi)^3 \cdot 2k^0} \left[\Theta(x^0-y^0)\, e^{-ik(x-y)} (\not{k}+m)_{\alpha\beta} - \Theta(y^0-x^0)\, e^{ik(x-y)} (\not{k}-m)_{\alpha\beta} \right]$$

and:

$$S_F\eta(x) = \int d^4y\, S_F(x-y)\eta(y), \quad \bar{\eta} S_F(x) = \int d^4y\, \bar{\eta}(y) S_F(y-x).$$

The full normalized correlator is then (generalized from 0+1 case):

$$\langle T\{ \psi_{\alpha_1}(x_1) \bar{\psi}_{\beta_1}(y_1) \psi_{\alpha_2}(x_2) \bar{\psi}_{\beta_2}(y_2) \dots \psi_{\alpha_n}(x_n) \bar{\psi}_{\beta_n}(y_n) \} \rangle$$

$$= \prod_{j=1}^{n} \left(-i \frac{\delta}{\delta \bar{\eta}_{\alpha_j}(x_j)} \right) \left(+i \frac{\delta}{\delta \eta_{\beta_j}(y_j)} \right) \exp\left(- \int d^4x\, d^4y\, \bar{\eta}(x) S_F(x-y) \eta(y) \right)$$

$$= \sum_{\sigma \in S_n} (-1)^{[\sigma]} \prod_{j=1}^{n} S_F(x_j - y_{\sigma(j)})_{\alpha_j \beta_{\sigma(j)}}$$