Noether's theorem and symmetries

Symmetries are profound tools in physics and linked to conserved quantities via Noether's theorem.

If a cont. symmetry exists, then under the shift $\varphi(x) \to \varphi(x) + \varepsilon f[\varphi(x)]$ where ε is infinitesimal and f represents a function of the field(s) ,

$\mathcal{L}$ is invariant up to $\mathcal{O}(\varepsilon^2)$. We will start by considering global symmetries, where ε is a constant, and later consider transformations that depend on coordinate x^μ and are thus <u>local</u>.

Consider $\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi$ for a scalar, real field φ. It is invariant under $\varphi \to \varphi + \varepsilon$, but if we added a mass term $-m^2\varphi^2$ it would not be.

Another example: $\mathcal{L}$ for complex, scalar field.

$$\mathcal{L} = \partial_\mu\varphi^*(x)\,\partial^\mu\varphi(x) - m^2\varphi^*(x)\varphi(x).$$

Under an infinitesimal shift $\varphi \to \varphi + i\varepsilon.\varphi(x)$, $\mathcal{L} \to \mathcal{L} + \mathcal{O}(\varepsilon^2)$.

This shows that a cont. symmetry exists: it is in fact

$\varphi(x) \to e^{i\theta}\varphi(x)$ which leaves $\mathcal{L}$ invariant and $\varphi \to \varphi + i\varepsilon.\varphi$ is the infinitesimal form of this transformation.

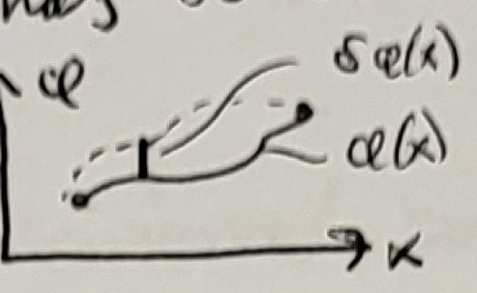
From classical mechanics, you know that Noether's theorem states that:

$\boxed{\text{For every continuous symmetry, there is a conserved current}}$

A symmetry transformation is supposed to leave the physics unchanged, i.e. the eqs. of mot. for the field. This is clearly the case if $\mathcal{L}$ is invariant, but it is also the case under more general circumstances: $\mathcal{L} \to \mathcal{L} + \varepsilon \partial_\mu J^\mu$ (*). Thus, if $\phi \to \phi + \delta\phi$ produces at most a divergence $\propto \partial_\mu J^\mu$ in $\mathcal{L} \to \mathcal{L} + \delta\mathcal{L}$, it is a symmetry transformation.

Recall that a variation is defined by first parametrizing the field by $\phi(\alpha, x)$ so that $\alpha = 0$ corresponds to the stationary value of S, and then:

$$\delta\phi \equiv \left.\frac{\partial \phi(\alpha, x)}{\partial \alpha}\right|_{\alpha=0} \cdot d\alpha$$



So which current is then conserved? Let $\phi \to \phi + \delta\phi$ (note that $\delta\phi$ can depend on both ϕ and $\partial_\mu\phi$) the latter for the case of translational invariance of $\mathcal{L}$ when $x_\mu \to x'_\mu$)

$$\mathcal{L} \to \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu\phi)$$

requires $\delta\phi$ to be infinitesimal

Now use that $\partial_\mu\phi = \partial_\mu\phi$ by definition [since $\phi = \phi(x)$] and also use that $\delta(\partial_\mu\phi) = \partial_\mu(\delta\phi)$ since the variation occurs at a fixed point x:
(i.e. derivative w.r.t. α and x commute)

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu\delta\phi + \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi$$

$$= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) - \underbrace{\left[\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} - \frac{\partial\mathcal{L}}{\partial\phi}\right]}_{=0}\delta\phi$$

$$= \varepsilon\,\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} f\right] = \varepsilon \cdot \partial_\mu J^\mu$$

if we have a symmetry transformation

(*) Easiest way to see why. Physical field configuration is the one that extremizes the action $S = \int L\,dt = \int \mathcal{L}\,d^4x$. We see that adding $\partial_\mu J$ gives $S = \ldots +$ surface term which we assume vanishes due to field vanishing at infinity and because there is no variation in the field at initial and final times. B it follows that $\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} f - \mathcal{J}^\mu \right) = 0$ and thus the current

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} f(\varphi) - \mathcal{J}^\mu \quad \text{is conserved.} \qquad (\text{recall } \delta\varphi \equiv \varepsilon \cdot f)$$

$f(\varphi)$ follows from the continuous transformation itself and $\mathcal{J}^\mu$ is identified from how $\mathcal{L}$ is changed. If $\delta\mathcal{L} = 0$, then $\mathcal{J}^\mu$ can be set to any constant.

Going back to our previous examples, we see that

- For $\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi$, the conserved current is $j^\mu = \partial^\mu\varphi(x)$ for $\varphi \to \varphi + \varepsilon$.
- For a complex field $\mathcal{L} = \partial_\mu\varphi^*(x)\,\partial^\mu\varphi(x) - m^2\varphi^*\varphi$, the conserved current is $j^\mu = i(\partial^\mu\varphi^*)\varphi - i\varphi^*\partial^\mu\varphi$ for $\varphi \to \varphi + i\varepsilon\varphi$. Note how j^μ is real

There are two terms in j^μ since we have to write

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\mu\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^*)}\partial_\mu\delta\varphi^* + \frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial\varphi^*}\delta\varphi^* .$$

We underline that the conserved current is identified by considering an infinitesimal transformation of the fields (we $\underset{\text{Taylor}}{\text{expand}}$ $\delta\mathcal{L}$ only to first order in $\delta\varphi$), but the obtained current is still conserved for a non-infinitesimal transformation corresponding to the infinitesimal one:

- The symmetry for $\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi$ is $\varphi \to \varphi + \text{const}$. The inf. form is $\varphi \to \varphi + \varepsilon$ which gives $j^\mu = \partial^\mu\varphi$, but j^μ is clearly conserved for $\varphi \to \varphi + \text{const}$. even if const. is not inf.

The symmetry for $\mathcal{L} = \frac{1}{2}\partial_\mu \varphi^* \partial^\mu \varphi - m^2 \varphi^* \varphi$ is $\varphi \to e^{i\theta}\varphi$.
The inf. form is $\varphi \to \varphi + i\varepsilon\varphi$ which gives $j^\mu = \ldots$, but this j^μ is clearly conserved for $\varphi \to \varphi e^{i\theta}$ even if θ is not inf.

We can understand this by noting that a non-inf. transformation can be built up by many inf. transformations, and the current is conserved in each step.

For the complex field, we see that the current is driven by the phase-gradient of the field: let $\varphi = \rho(x) e^{i\theta(x)}$, then

$$j^\mu = 2\rho^2(x)\, \partial^\mu \theta(x).$$

The time-component of the current is the charge density*, so we may construct the charge as follows.

$$Q = \int d^3x \; j^0(\vec{x}, x^0)$$

This is constant in time so long that $\vec{j}(x) \to 0$ for $x \to \infty$:

$$\frac{\partial Q}{\partial t} = \int d^3x \frac{\partial j^0(\vec{x}, x^0)}{\partial t} = -\int d^3x \, \nabla \cdot \vec{j}(\vec{x}, x^0) = -\int_\infty d\vec{S} \cdot \vec{j} = 0$$

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* For an electric current, j^0 is the physical charge density. Here, we use the word in a generalized way: currents carrying different properties have some associated charge