3.1.2 Time-energy uncertainty: what it *really* means

Whereas the preceeding uncertainty relation $\Delta x \Delta p_x \geq \hbar/2$ is well-defined mathematically and in terms of physical meaning, one often encounters a similar uncertainty relation involving time and energy in the literature. It has the form

$$\Delta E \Delta t \geq \hbar/2. \tag{3.2}$$

The problem with this relation is that it is not immediately clear how it is obtained or what it even means. For instance, while there is an energy operator in QM (the Hamiltonian $\hat{H}$), *there is no time operator* $\hat{T}$ in QM. In fact, time is an independent variable in non-relativistic QM. Thus, from the very outset it is clear that we cannot derive Eq. (3.2) in the same way as Eq. (3.1), the latter requiring well-defined position and momentum operators. Moreover, what does Δt even mean? This has caused a lot of discussion regarding the existence and meaning of Eq. (3.2) and, unfortunately, some gross misinterpretations of it. Here, we will show (as was originally done in Ref. [30]) that Eq. (3.2) can be derived in non-relativistic QM as long as one properly defines the meaning of ΔE and Δt. Then, we will discuss its physical interpretation. We note in passing that one can also derive uncertainty principles of similar form as the well-known non-relativistic ones, albeit with relativistic corrections.

Before proceeding, let us briefly remind ourselves of the difference between stationary and non-stationary states. A stationary state Ψ_n solves both the time-dependent SE and is an eigenfunction of a time-independent H at the same time. It can be written as $\Psi_n = \psi_n \mathrm{e}^{-\mathrm{i}E_n t/\hbar}$ where ψ is an eigenfunction of H while E is the belonging eigenenergy. If the system is in a stationary state, all expectation values of observable quantities are time-independent.

Now, according to the superposition principle we also know that any linear combination of stationary states will also be a solution to the time-dependent SE, so that a general physical state of the system (still for a time-independent Hamiltonian) may be written as

$$\Psi = \sum_n c_n \Psi_n = \sum_n c_n \psi_n \mathrm{e}^{-\mathrm{i}E_n t/\hbar}. \tag{3.3}$$

The key point is that Ψ is no longer an eigenfunction of $\hat{H}$, meaning that Ψ is not a stationary state. Both the probability density $|\Psi|^2$ and the expectation value of physical observables may now depend on time, and energy is not sharply defined anymore ($\Delta E \neq 0$).

Let us now sketch the proof of Eq. (3.2). Let H be a time-independent Hamiltonian and let Ψ be the wavefunction of the system (but not necessarily stationary). In that case, we can show that in the Heisenberg picture (where the time-dependence is placed on the operators whereas the wavefunctions are time-independent), the expectation value of a quantity A that does not depend on time ($\partial A/\partial t = 0$) explicitly satisfies:

$$\frac{d}{dt}\langle A \rangle = \frac{1}{\mathrm{i}\hbar}\langle [A, H] \rangle \tag{3.4}$$

Let ΔA and ΔE denote the root-mean-square deviations (also known as standard deviations) of A and H, respectively:

$$\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}. \tag{3.5}$$

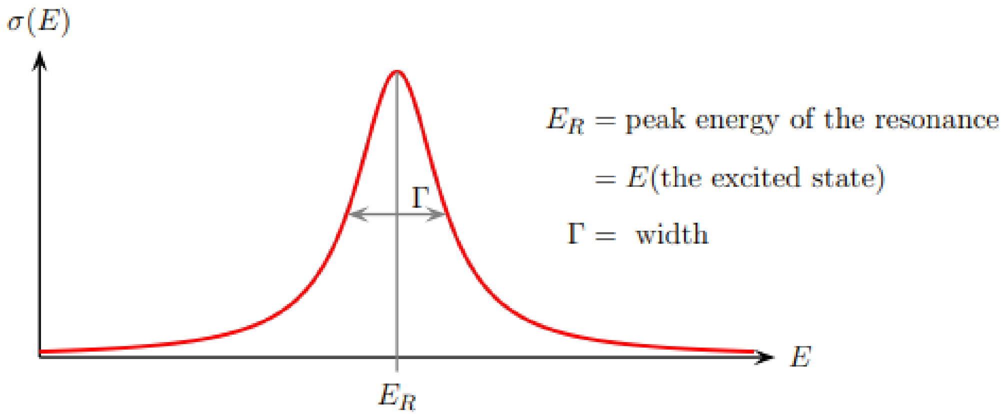

Figure 3.1: Sketch of energy-dependence of scattering cross section (or the number of detected events of a particular reaction) with a peak and corresponding linewidth that is inversely proportional to the lifetime of the decaying particle. Figure taken from `https://web2.ph.utexas.edu/~vadim/Classes/2019f/resonances.pdf`.

One can then show that (see *e.g.* chapter 3 in Griffiths QM book) $\Delta A \cdot \Delta E \geq \frac{1}{2}|\langle[A, H]\rangle|$ using these definitions. Inserting our above expression for the commutator between A and H, one obtains:

$$\Delta E \cdot \frac{\Delta A}{|d\langle A\rangle/dt|} \geq \hbar/2. \tag{3.6}$$

which may be written precisely as

$$\Delta E \Delta t \geq \hbar/2, \tag{3.7}$$

if we define

$$\Delta t \equiv \frac{\Delta A}{|d\langle A\rangle/dt|}. \tag{3.8}$$

The crucial point regarding the physical interpretation of this uncertainty relation between ΔE and Δt is to recognize what Δt means. From its definition above, we see that Δt is the time required for the expectation value of A to change by an amount equal to its standard deviation ΔA. Put in more informal terms, it is the time required for the expectation value of A to change appreciably (with "appreciably" quantitatively being defined by the standard deviation).

If the system is in a stationary state, then we know that $d\langle A\rangle/dt = 0$ so that $\Delta t \to \infty$, but that is perfectly fine since $\Delta E \to 0$ then and the inequality is still valid. For a non-stationary state, however, $\Delta E \neq 0$ is the standard deviation of the Hamiltonian H and Δt can be thought of as the lifetime of the state Ψ with respect to the observable A, according to our above explanation. More precisely, it is the time interval after which the expectation value of A has changed appreciably (as defined via the standard deviation of A).

How do we interpret this physically? One consequence is that a state that exists only for a short time cannot have a well-defined energy. For instance, an excited state in a condensed matter system that has a finite lifetime will then release a slightly different energy each time it decays, and the spread in this energy will be larger (meaning larger ΔE) the shorter its lifetime Δt. For a long-lived excitation $\Delta t \to \infty$, energy becomes well-defined $\Delta E \to 0$. This uncertainty in energy is reflected in the natural linewidth of the distribution of energies released by state that has decayed in this manner: fast-decaying states have a broad linewidth. The same principle also applies to fast-decaying particles in particle physics: the faster the particle decays, the shorter its lifetime and the less certain is its mass. We note that to detect particles with long lifetimes, one can simply observe the distance they propagate before decaying, which historically was done using *e.g.* bubble chambers. This no longer becomes possible for particles with lifetimes as short as for instance $\tau \to 10^{-20}$ s decaying via the strong interaction, since they propagate extremely short distances before disintegrating.

The above reasoning seems to suggest that there should be a strong link between the time-energy uncertainty relation and the concept of *quantum fluctuations*. A well-known example of a quantum fluctuation in particle physics is the bubble polarization diagram where a photon is converted into a temporary electron-positron pair which then collapses back into a photon. The fact that there exists confusing statements in the literature, such as that this can happen because the $e^- - e^+$ pair ”borrows” energy from the environment, just underlines the importance of correctly interpreting what the time-energy uncertainty relation means.

In light of the explanation we've given above, we can now understand that the reason that such spontaneous particle pairs can occur as a quantum fluctuation is that the energy of vacuum cannot be sharply defined/accurately determined. In a cartoon picture, one can think of the vacuum fields ”jittering” constantly and thus have what is referred to as a zero-point energy, the latter statement really just expressing that $\Delta E \neq 0$. Now, because of this uncertainty in the energy of vacuum, we are allowed to create spontaneous pairs corresponding to the bubble diagram in particle physics that exist a finite time Δt as long as $\Delta E \Delta t \geq \hbar/2$. Please note the important distinction between saying that energy is not sharply defined, which is a statistical statement, and saying that energy can appear out of nothing or being ”borrowed” from some ill-defined environment. Vacuum is not a stationary state, because in that case we would have $\Delta E = 0$. Note that a Hamiltonian describing vacuum can still be time-independent even if it is non-stationary.