5.2.1 Green's function

The Green's function we seek is a solution $D(z^\rho)$ of $\partial_\mu \partial^\mu D(z^\rho) = \delta^4(z^\rho)$ where $z^\rho = x^\rho - x'^\rho$. Appealing to Fourier transforms again, we have

$$\partial_\mu \partial^\mu \frac{1}{(2\pi)^2} \int d^4k \, \tilde{D}(k) e^{-ik_\nu z^\nu} = -\frac{1}{(2\pi)^2} \int d^4k \, k_\mu k^\mu \tilde{D}(k^\mu) e^{-ik_\nu z^\nu} = \frac{1}{(2\pi)^4} \int d^4k \, e^{-ik_\nu z^\nu}$$

from which

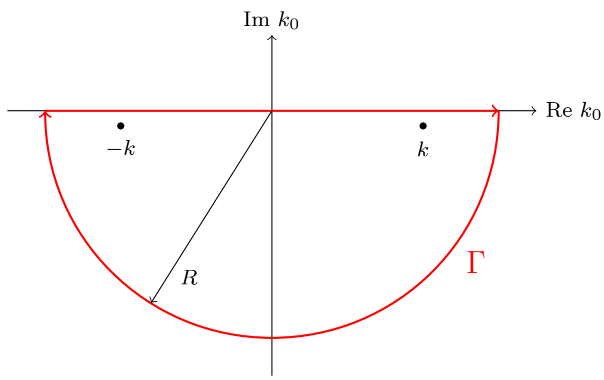
$$\tilde{D}(k^\mu) = -\frac{1}{4\pi^2} \frac{1}{k_\mu k^\mu} = -\frac{1}{4\pi} \frac{1}{k_0^2 - k^2}$$

where $k = |\vec{k}|$. We now must solve

$$D(z^\mu) = -\frac{1}{(2\pi)^4} \int d^4k \, \frac{1}{k_0^2 - k^2} e^{-ik_\nu z^\nu} = -\frac{1}{(2\pi)^4} \int d^3k \, e^{i\vec{k}\cdot\vec{z}} \int_{-\infty}^{\infty} dk_0 \frac{1}{k_0^2 - k^2} e^{-ik_0 z^0}$$

Note the sign change in the first exponential when converting to three-dimensions. Now, to evaluate the k_0 integral we use contour integration, treating k_0 as a complex number, $k_0 = \mathrm{Re}\, k_0 + i \mathrm{Im}\, k_0$. This gives $e^{i\vec{k}\cdot\vec{z}} = e^{\mathrm{Im}\, k_0 z^0} e^{-i \mathrm{Re}\, k_0}$. So that the integral converges we must choose $\mathrm{Im}\, k_0 < 0$ for $z^0 > 0$. This condition is imposed as $z^0 = x^0 - x'^0 = c(t - t')$, and so positive z^0 ensures that contributions to the Green's function and hence to the potential A^μ only come from events that occur at times $t' < t$, i.e. events in the past. Thus causality is ensured.

Now, the poles of the integrand are $\pm k$ and so lie on the real axis. To avoid them, we displace our contour by an infinitesimal amount $i\epsilon$ so that it lies just in the upper-half plane (formally we should let $\epsilon \to 0$ at the end). Our contour of integration Γ then looks like:



We than have

$$\oint_\Gamma dk_0 \frac{e^{-ik_0 z^0}}{k_0^2 - k^2} = \int_{-R}^{R} dk_0 \frac{e^{-ik_0 z^0}}{k_0^2 - k^2} + \int_{\text{semi-circle}} \frac{e^{-ik_0 z^0}}{k_0^2 - k^2}$$

Now on the semicircle we can write $k_0 = Re^{-i\varphi} = R\cos\varphi - iR\sin\varphi$, so that the integral becomes an integral over φ from 0 to π. So we have

$$\int_{\text{semi-circle}} \frac{e^{-ik_0 z^0}}{k_0^2 - k^2} = -i \int_0^\pi d\varphi \, Re^{i\varphi} e^{-Rz^0 \sin\varphi - iRz^0 \cos\varphi} \frac{1}{R^2 e^{-2i\varphi} - k^2}$$

and $\sin\varphi \geq 0$ for this range of φ, hence the integrand goes to zero as $R \to \infty$, as required. The value of the k_0 integral we are interested in will hence be given by the $-2\pi i$ times the sum of the residues inside the contour. The residues are:

$$\lim_{k_0 \to k}(k_0 - k)\frac{e^{-ik_0 z^0}}{(k_0-k)(k_0+k)} = \frac{e^{-ikz^0}}{2k}$$

$$\lim_{k_0 \to -k}(k_0 + k)\frac{e^{-ik_0 z^0}}{(k_0-k)(k_0+k)} = -\frac{e^{ikz^0}}{2k}$$

hence

$$\int_{-\infty}^{\infty} dk_0 \frac{1}{k_0^2 - k^2} e^{-ik_0 z^0} = \frac{\pi i}{k}\left(e^{ikz^0} - e^{-ikz^0}\right)\Theta(z^0)$$

where the Heaviside function

$$\Theta(z^0) = \begin{cases} 1 & z^0 > 0 \\ 0 & z^0 < 0 \end{cases}$$

is added as the residues only contribute for positive z^0. So we now have

$$\begin{aligned} D(z^\mu) &= \frac{\Theta(z^0)}{(2\pi)^4}\pi i \int d^3k\, e^{i\vec{k}\cdot\vec{z}} \frac{1}{k}\left(e^{ikz^0} - e^{-ikz^0}\right) \\ &= \frac{\Theta(z^0)}{16\pi^3} i \int_0^\infty dk\, k^2 \frac{1}{k}\left(e^{ikz^0} - e^{-ikz^0}\right)\int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta\, e^{ikz\cos\theta} \end{aligned}$$

upon switching to polar coordinates and choosing the coordinate frame such that the k^3 axis makes an angle of θ with $\vec{z}$, and letting $z = |\vec{z}|$. Integrating over the angles, we get

$$\begin{aligned} D(z^\mu) &= \frac{\Theta(z^0)}{8\pi^2} i \int_0^\infty dk\, k\left(e^{ikz^0} - e^{-ikz^0}\right)\left(\frac{e^{ikz}}{ikz} - \frac{e^{-ikz}}{ikz}\right) \\ &= -\frac{\Theta(z^0)}{8\pi^2 z}\int_0^\infty dk\left(e^{ik(z^0+z)} + e^{-ik(z^0+z)} - e^{ik(z^0-z)} - e^{-ik(z^0-z)}\right) \end{aligned}$$

but if we let $k \mapsto -k$

$$\int_0^\infty dk\, e^{-ik(z^0+z)} = \int_0^{-\infty} d(-k) e^{ik(z^0+z)} = \int_{-\infty}^0 dk\, e^{ik(z^0+z)}$$

hence

$$D(z^\mu) = -\frac{\Theta(z^0)}{8\pi^2 z}\int_{-\infty}^\infty dk\left(e^{ik(z+z^0)} - e^{ik(z^0-z)}\right) = \frac{\Theta(z^0)}{4\pi z}\left(\delta(z^0 - z) - \delta(z^0 + z)\right)$$

remembering the integral representation of the delta function. Owing to the Heaviside function only the first delta function will contribute, so

$$D_{ret}(z^\mu) = \frac{\Theta(z^0)}{4\pi z}\delta(z^0 - z)$$

The subscript signifies that this is the *retarded* Green's function (that is, the Green's function resulting from the contribution of events in the past).

We can put the Green's function in covariant form using

$$\delta(z_\mu z^\mu) = \delta([z_0 - z][z_0 + z]) = \frac{\delta(z_0 - z)}{|z_0 + z|} + \frac{\delta(z_0 + z)}{|z_0 - z|}$$

as $\delta(ab) = \frac{\delta(a)}{|b|} + \frac{\delta(b)}{|a|}$. Hence

$$\Theta(z^0)\delta(z_\mu z^\mu) = \Theta(z^0)\frac{\delta(z_0 - z)}{|z_0 + z|} = \Theta(z^0)\frac{\delta(z_0 - z)}{|2z|}$$

and so

$$D_{ret}(z^\mu) = \frac{\Theta(z^0)}{2\pi}\delta(z_\mu z^\mu)$$

Recalling that $z^\mu = x^\mu - x'^\mu$ we can state our final results for the Green's function as

$$D_{ret}(x^\mu - x'^\mu) = \frac{\Theta(x^0 - x'^0)}{4\pi z}\delta(x^0 - x'^0 - |\vec{x} - \vec{x'}|)$$

and in covariant form,

$$D_{ret}(x^\mu - x'^\mu) = \frac{\Theta(x^0 - x'^0)}{2\pi}\delta([x_\mu - x'_\mu][x^\mu - x'^\mu])$$

Note that if we had taken $z^0 < 0$ and closed our contour in the upper half-plane (with poles displaced upwards) we would have obtained the *advanced* Green's function

$$D_{adv}(x^\mu - x'^\mu) = \frac{\Theta(x'^0 - x^0)}{4\pi z}\delta(x^0 - x'^0 + |\vec{x} - \vec{x'}|)$$

5.2.2 Lienard-Wiechart potentials

Of night and light and the half-light.

W.B. Yeats, "He Wishes For The Cloths Of Heaven"

We can now work out the potentials A^μ that solve the Maxwell equation $\partial_\mu \partial^\mu A^\nu = \frac{4\pi}{c} J^\nu$. They are given by

$$A^\mu(x^\sigma) = \frac{4\pi}{c}\int d^4x' D_{ret}(x - x')J^\mu(x'^\sigma)$$

where

$$J^\mu(x'^\sigma) = e\frac{dx'^\sigma}{dt'}\delta^3(\vec{x}' - \vec{x}'_e(t))$$

or in covariant form

$$J^\mu(x'^\sigma) = ec\int d\tau\frac{dx'^\sigma}{d\tau}\delta^4(x'^\sigma - x'^\sigma_e(\tau))$$