Now, the difference between contravariant $x^\mu = (t, x, y, z)$ and co~~ntravariant~~ vector (co is lo) $x_\mu = (t, -x, -y, -z)$ is how the two transform under a change of basis vectors: the contravariant one does generally not transform like the basis vectors, while the covariant does. In Euclidean space, this is like the difference between $\vec{r}$ and ∇.

EXAMPLE

Let $\hat{x}$ and $\hat{y}$ be basis vectors and define a new basis $\hat{x}'$ and $\hat{y}'$: $\hat{x}' = B^1{}_1 \hat{x} + B^2{}_1 \hat{y}$, $\hat{y}' = B^1{}_2 \hat{x} + B^2{}_2 \hat{y}$. May write compactly as

$$\begin{pmatrix} \hat{x}' \\ \hat{y}' \end{pmatrix} = \begin{pmatrix} B^1{}_1 & B^2{}_1 \\ B^1{}_2 & B^2{}_2 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \equiv B \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}.$$

Then, it follows that the new components x' and y' of a vector $\vec{v} = x\hat{x} + y\hat{y}$ in the original basis, written $\vec{v} = x'\hat{x}' + y'\hat{y}'$ in the new basis, are given by:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = (B^T)^{-1} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Note that $(B^T)^{-1} = (B^{-1})^T$ for an invertible matrix.

Now, consider instead the gradient operator $\nabla = (\partial_x, \partial_y)$.

The chain rule gives: $\frac{\partial}{\partial x'} = \frac{\partial}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial}{\partial y}\frac{\partial y}{\partial x'}$ and similarly for $\frac{\partial}{\partial y'}$. We may compute $\frac{\partial x}{\partial x'}$, $\frac{\partial y}{\partial x'}$, etc. since we have $\begin{pmatrix} x \\ y \end{pmatrix} = B^T \begin{pmatrix} x' \\ y' \end{pmatrix}$. Doing so provides:

$$\begin{pmatrix} \frac{\partial}{\partial x'} \\ \frac{\partial}{\partial y'} \end{pmatrix} = B \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}.$$

Thus, it follows that the components of ∇ transform just like the basis vectors themselves, while the components of $\vec{r}$ transform differently in general. Only in the case of a pure rotation (where $(B^{-1})^T = B$) would the transformation be the same.

In contrast, if the basis change corresponds to a <u>scaling</u> (diagonal B) then $B = B^T$, but $B \neq B^{-1}$, so the contravariant and covariant vectors transform oppositely.