FY3403 Particle physics
Problemset Extra fall 2024

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Problem 1. A second look at isospin

Particles are introduced into the standard model by placing quantum fields in irreducible representations of the standard model symmetry group $g_{sm} = SU(3) \times SU(2) \times U(1)$. This exercise is *not* meant to introduce all the machinery necessary to understand such a statement, but to motivate the use of representation theory in particle physics.

Representation theory is all about representing objects that are abstractly challenging (e.g. groups) with matrices acting on vectors. Representation theory is a vast and ubiquitous subject studied by mathematicians and physicists alike. We shall, however, showcase its use in phenomenological particle physics by the means of a toy model. Groups arise in physics as symmetries, and we shall look into the treatment of the (approximate) symmetry isospin.

Isospin conservation stems from the invariance of some physical processes under the action $u \to d$ and $d \to u$. We model this, however, not as a discrete symmetry (like i.e. parity), but as a continuous $G = SU(2)$ symmetry. It is important to realise that what we are actually doing is guessing on a group (G) and comparing the predictions following from that guess with the actual world. Had they not matched, the initial guess would have been wrong. One prediction we should find is the existence of two particles (up and down quark) which are equivalent to each other (at least as far as this symmetry is concerned).

Think of an element $g \in G = SU(2)$ as some abstract entity which can be *represented* as a matrix $M_g \in \mathbb{C}^{2\times 2}$ with unit determinant. The map $g \to M_g$ must obey the group structure of G, and we therefore require

$$g_1 \cdot g_2 = g_3 \leftrightarrow M_{g_1} @ M_{g_2} = M_{g_3} \tag{1}$$

where the @ on the right is regular matrix multiplication. The map $g \to M_g$ is what mathematicians actually call a representation, because you are representing the abstract elements $g \in G$ by the much more hands-on matrices $M_g \in \mathbb{C}^{2\times 2}$ in a product-preserving way. The representation is 2-dimensional, because the vector space on which the matrices act is 2-dimensional.

a) The M_g matrices act on a 2-dimensional Hilbert space $\mathcal{H}$. Let $|u\rangle$ and $|d\rangle$ be two orthonormal vectors in $\mathcal{H}$. Show that for all $g \in G$, the inner products in this subspace is unaffected by the transformation M_g. (*Hint:* think of $|u\rangle$ and $|d\rangle$ as unit vectors in Euclidean space)

b) If we assume the representation $g \to M_g$ combined with the Hilbert space $\mathcal{H}$ is a suitable model of the universe, then, as all observables of quantum mechanics are given by inner products as above (see the zeroth axiom of the Wightman axioms), quantum mechanics should be invariant under the action of $SU(2)$. If we now name $|u\rangle$ as 'up-quark' and

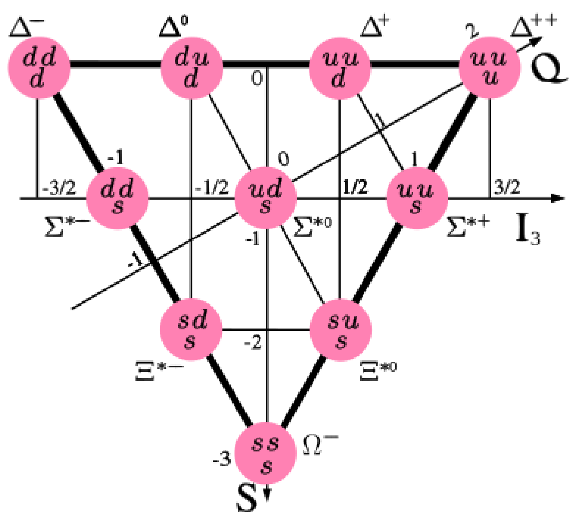
$|d\rangle$ as 'down-quark', these assumptions have predicted the existence of two particles. This begs the question: does there exist higher dimensional representations of $SU(2)$? The answer is 'yes' and you have actually seen them in quantum mechanics when you worked with angular momentum.

The (irreducible[1]) representations R_l of $SU(2)$ are labeled by the total angular momentum quantum number l. Their dimensions d_l are the number of different angular momentum eigenstates sharing the same l. For $l = 0, \frac{1}{2}, 1, \frac{3}{2}$, what is d_l?

c) In physics nomenclature, we say that we place a particle field in the R_l representation of $SU(2)$. What this means is that we postulate the existence of d_l particles transforming into each other under the action of $SU(2)$ and that all of these particles are (from the symmetry point of view) the same. With this language, we would rephrase our previous prediction (the up and down quarks) by saying the universe has a particle field in the $R_{\frac{1}{2}}$ representation of $SU(2)$.

Consider Fig. 1. Analogously to what we did with u and d above, in what representations R_l of $SU(2)$ would you place the Ω, Ξ, Σ and Δ baryons?

Figure 1: The baryon decuplet (figure from Wikipedia).



d) Which representation R_l would you choose to model the pion?

e) Why are there no particles in the universe represented by R_2?

Problem 2. Spontaneous symmetry breaking (SSB)

This exercise is meant to illustrate the important mechanism of *spontaneous symmetry breaking*, where a symmetric potential can lead to ground states where the symmetry is in some sense 'lost'.

[1] It is important that the representations are irreducible. See for example here

Consider a classical particle parameterized by the two-dimensional coordinate $r = (x(t), y(t))$, and let it travel according to the following Hamiltonian

$$H = \frac{1}{2}(m\dot{x}^2 + \dot{y}^2) - V(x, y) \tag{2}$$

where $V(x, y) = a(x^2 + y^2) + b(x^2 + y^2)^2 = a(x^2 + y^2) + b(x^4 + 2x^2y^2 + y^4)$

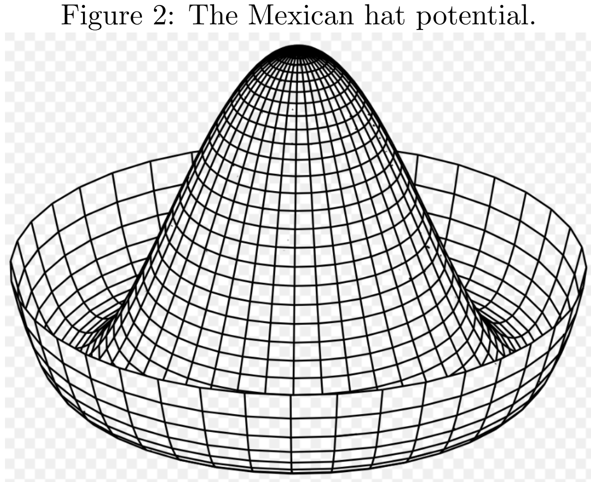
a) What symmetries does the potential posses?

b) Consider the case $a, b > 0$. What is the minimum of the potential? At which point does this minimum occur?

c) Suppose a particle is situated at the minimum (x_{min}, y_{min}). A way to check if this minimum is stable is to change to polar coordinates centered at (x_{min}, y_{min}) and calculate how the potential looks like locally around $r = 0$. Taylor expand $V(x, y)$ around (x_{min}, y_{min}) to second order and express the result in polar coordinates. Argue whether the minima are stable or not.

d) Consider now the case $a < 0 < b$. What is the minimal value now? At which point(s) does this minimal value occur? (*Hint:* the potential is drawn for this case in Fig. 2.)

Figure 2: The Mexican hat potential.



e) Pick one of the points where the minimal value occur and do the Taylor expansion as above. Argue whether your minima is stable or not. (*Hint:* remember to use polar coordinates centered at the point you chose.)

f) If you have done all the above exercises correctly, you will have observed that the ground state in c is stable and retains the original symmetry of the potential, whereas the ground state(s) in d does not. At what point in your calculation in d do you actually lose the symmetry of the initial problem?

Problem 3. Mass generation from SSB

A full treatment of the Higgs mechanism in the Standard model requires writing down a formidable Lagrangian taking into account interactions between many different fields. This exercise is instead meant to show the basic mechanism by which mass terms can be modified in a Lagrangian when symmetries are broken.

Consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - V(x), V(x) = -ax^2 + bx^4 \tag{3}$$

where $a, b > 0$

a) Complete the square in the potential term and discard any constants.

b) Introduce the variable $y = x^2 - \frac{a}{2b}$ and write $\mathcal{L}$ as a function of y.

c) Assume y is small and Taylor expand to leading order. What is the mass term in this new Lagrangian?