FY3403 Particle physics
Problemset Extra

NTNU

Institutt for fysikk

SUGGESTED SOLUTION

Problem 1
Particles are introduced into the standard model by placing quantum fields in irreducible representations of the standard model symmetry group $g_{sm} = SU(3)\times SU(2)\times U(1)$. This exercise is *not* meant to introduce all the machinery necessary to understand such a statement, but to motivate the use of representation theory in particle physics.

Representation theory is all about representing objects that are abstractly challenging (e.g. groups) with matrices acting on vectors. Representation theory is a vast and ubiquitous subject studied by mathematicians and physicists alike. We shall, however, showcase its use in phenomenological particle physics by the means of a toy model. Groups arise in physics as symmetries, and we shall look into the treatment of the (approximate) symmetry isospin.

Isospin conservation stems from the invariance of some physical processes under the action $u \to d$ and $d \to u$. We model this, however, not as a discrete symmetry (like i.e. parity), but as a continuous $G = SU(2)$ symmetry. It is important to realise that what we are actually doing is guessing on a group (G) and comparing the predictions following from that guess with the actual world. Had they not matched, the initial guess would have been wrong. One prediction we should find is the existence of two particles (up and down quark) which are equivalent to each other (at least as far as this symmetry is concerned).

Think of an element $g \in G = SU(2)$ as some abstract entity which can be *represented* as a matrix $M_g \in \mathbb{C}^{2\times 2}$ with unit determinant. The map $g \to M_g$ must obey the group structure of G, and we therefore require

$$g_1 \cdot g_2 = g_3 \leftrightarrow M_{g_1} @ M_{g_2} = M_{g_3} \tag{1}$$

where the @ on the right is regular matrix multiplication. The map $g \to M_g$ is what mathematicians actually call a representation, because you are representing the abstract elements $g \in G$ by the much more hands-on matrices $M_g \in \mathbb{C}^{2\times 2}$ in a product-preserving way. The representation is 2-dimensional, because the vector space on which the matrices act is 2-dimensional.

a) The M_g matrices act on a 2-dimensional Hilbert space $\mathcal{H}$. Let $|u\rangle$ and $|d\rangle$ be two orthonormal vectors in $\mathcal{H}$. Show that for all $g \in G$, the inner products in this subspace is unaffected by the transformation M_g. (*Hint:* think of $|u\rangle$ and $|d\rangle$ as unit vectors in Euclidean space)

SOLUTION: Set

$$u = \begin{pmatrix}1\\0\end{pmatrix}\, d = \begin{pmatrix}0\\1\end{pmatrix} \tag{2}$$

Any $|\psi\rangle = a|u\rangle + b|v\rangle$ can therefore be parameterized by the 2-dimensional vector $\begin{pmatrix}a & b\end{pmatrix}$. The matrices M_g for $g \in G$ induce the transformation

$$|\psi\rangle \to |\psi'\rangle = M_g |\psi\rangle \tag{3}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} \to M_g \begin{pmatrix} a \\ b \end{pmatrix} \tag{4}$$

Let $|\phi\rangle = \alpha |u\rangle + \beta |d\rangle$. Then

$$\langle \psi' | \phi' \rangle = \langle \psi | M_g^\dagger M_g |\phi\rangle = \langle \psi | |\phi\rangle \tag{5}$$

because $M_g^\dagger M_g = I_{2\times 2}$.

b) If we assume the representation $g \to M_g$ combined with the Hilbert space $\mathcal{H}$ is a suitable model of the universe, then, as all observables of quantum mechanics are given by inner products as above (see the zeroth axiom of the Wightman axioms), quantum mechanics should be invariant under the action of $SU(2)$. If we now name $|u\rangle$ as 'up-quark' and $|d\rangle$ as 'down-quark', these assumptions have predicted the existence of two particles. This begs the question: does there exist higher dimensional representations of $SU(2)$? The answer is 'yes' and you have actually seen them in quantum mechanics when you worked with angular momentum.

The (irreducible[1]) representations R_l of $SU(2)$ are labeled by the total angular momentum quantum number l. Their dimensions d_l are the number of different angular momentum eigenstates sharing the same l. For $l = 0, \frac{1}{2}, 1, \frac{3}{2}$, what is d_l?

SOLUTION: The number of different total angular momentum eigenstates $|l,m\rangle$ follows from standard quantum mechanics and we get $d_l = 2 \cdot l + 1$. Thus $d_0 = 1, d_{\frac{1}{2}} = 2, d_1 = 3$ and $d_{\frac{3}{2}} = 4$.

c) In physics nomenclature, we say that we place a particle field in the R_l representation of $SU(2)$. What this means is that we postulate the existence of d_l particles transforming into each other under the action of $SU(2)$ and that all of these particles are (from the symmetry point of view) the same. With this language, we would rephrase our previous prediction (the up and down quarks) by saying the universe has a particle field in the $R_{\frac{1}{2}}$ representation of $SU(2)$.

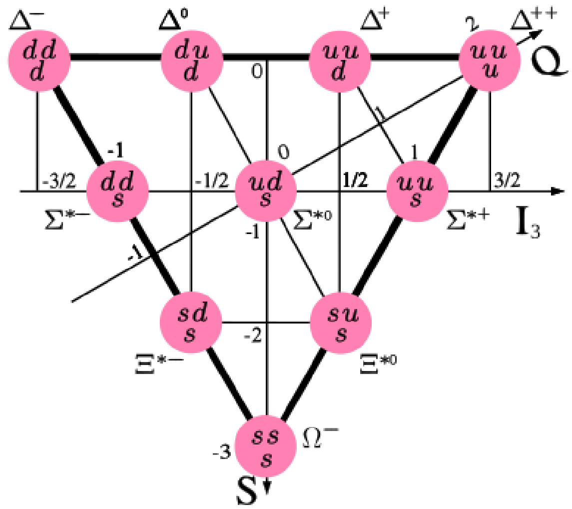
Consider figure 1. Analogously to what we did with u and d above, in what representations R_l of $SU(2)$ would you place the Ω, Ξ, Σ and Δ baryons?

SOLUTION: This is a question of number matching. There is only one Ω baryon, and we would therefore pick R_0 (being the only representation of dimension 1). Similarly, the Δ baryon would be placed in $R_{\frac{3}{2}}$, because $d_{\frac{3}{2}} = 4$. Following this logic, we end up with

$$\Omega \in R_1$$
$$\Xi \in R_{\frac{1}{2}}$$
$$\Sigma \in R_1$$
$$\Delta \in R_{\frac{3}{2}}$$

[1] It is important that the representations are irreducible. See for example here

Figure 1: The baryon decuplet (figure from Wikipedia).



We assumed that there exists particles in these representations, and nature provided us with the baryons. In quantum field theory, the (say) Δ baryon would be modeled as a 4-dimensional field whose components transform into each other under the action of $SU(2)$. Any interaction term that only cares about the four-dimensional field Δ as a whole and *not* its individual components, would then preserve the isospin structure.

d) Which representation R_l would you choose to model the pion?

SOLUTION: The 3 different pions have total isospin $l = 1$ and are therefore (isospin-wise) equal to the Σ. Thus I would use R_1.

e) Why are there no particles in the universe represented by R_2?

SOLUTION: This is rather a trick question and starts on the wrong side of how physics works. Going through the toy model above, we did not at any point use the fact that nature has given us some particles. As theorists, we guessed that there are particles in the four lowest representation of $SU(2)$ and showed that nature actually has given us these particles. Had one of the particles been missing, say Σ^-, then our model would be wrong.

The standard model does not give any answer to the question *why these specific representations*[2]. It states as an axiom the representations concerned and show that it is a good match of nature. This is one of the absolute main drawbacks of the standard model.

Problem 2

[2]This is actually one of the maingoals of Grand Unified Theories

This exercise is meant to illustrate the important mechanism of *spontaneous symmetry breaking*, where a symmetric potential can lead to ground states where the symmetry is in some sense 'lost'.

Consider a classical particle parameterized by the two-dimensional coordinate $r = (x(t), y(t))$, and let it travel according to the following Hamiltonian

$$H = \frac{1}{2}(m\dot{x}^2 + \dot{y}^2) - V(x,y) \tag{6}$$

where $V(x,y) = a(x^2+y^2) + b(x^2+y^2)^2 = a(x^2+y^2) + b(x^4+2x^2y^2+y^4)$.

a) What symmetries does the potential posses?

SOLUTION: Rotationally symmetric around the z-axis.

b) Consider the case $a, b > 0$. What is the minimum of the potential? At which point does this minimum occur?

SOLUTION: if a and b are positive, then the potential is nonnegative. This implies $x = y = 0$ is the only minimum of the potential with $V(0,0) = 0$.

c) Suppose a particle is situated at the minimum (x_{min}, y_{min}). A way to check if this minimum is stable is to change to polar coordinates centered at (x_{min}, y_{min}) and calculate how the potential looks like locally around $r = 0$. Taylor expand $V(x,y)$ around (x_{min}, y_{min}) to second order and express the result in polar coordinates. Argue whether the minima are stable or not.

SOLUTION: As we are using polar coordinates centered at the origin, this is just regular polar coordinates. Let $\boldsymbol{x} = (x,y)$ and $\boldsymbol{x}_0 = (x_{min}, y_{min})$

$$V(x,y) \approx V(\boldsymbol{x}_0) + \nabla V(\boldsymbol{x}_0)(\boldsymbol{x}-\boldsymbol{x}_0) + \frac{1}{2}(\boldsymbol{x}-\boldsymbol{x}_0)^T H(\boldsymbol{x}_0)(\boldsymbol{x}-\boldsymbol{x}_0). \tag{7}$$

We thus need the gradient and the hessian of the potential, which are

$$\nabla V = \begin{pmatrix} 2ax + 4bx(x^2+y^2) \\ 2ay + 4by(x^2+y^2) \end{pmatrix}, \; H = \begin{pmatrix} 2a + 8bx^2 + 4b(x^2+y^2) & 8bxy \\ 8bxy & 2a + 8by^2 + 4b(x^2+y^2) \end{pmatrix} \tag{8}$$

which at $(0,0)$ gives

$$\nabla V(0,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} H(0,0) = \begin{pmatrix} 2a & 0 \\ 0 & 2a \end{pmatrix}. \tag{9}$$

We thus end up with

$$V(x,y) \approx a(x^2+y^2) = ar^2 \tag{10}$$

which, of course, match the potential above. The potential is stable, since any perturbation $r > 0$ would lead to an increase in the energy.

d) Consider now the case $a < 0 < b$. What is the minimal value now? At which point(s) does this minimal value occur?

SOLUTION:

$$V(r) = ar^2 + br^4 = b(r^4 + \frac{a}{b}r^2) = b(r^2 + \frac{a}{2b})^2 - \frac{a^2}{4b} \tag{11}$$

has the minimal value

$$-\frac{a^2}{4b} \tag{12}$$

when

$$r^2 = -\frac{a}{2b}, \tag{13}$$

that is for all values x and y on the circle of radius $-\frac{a}{2b}$.

e) Pick one of the points where the minimal value occur and do the Taylor expansion as above. Argue whether your minima is stable or not. (*Hint:* remember to use polar coordinates centered at the point you chose)

SOLUTION: The calculation of the derivative is the same as above. The only difference is the base point of the polar coordinates, which I choose to be $(\sqrt{-\frac{a}{2b}}, 0)$ on the positive x-axis. From the derivatives above, one obtains

$$\nabla V(\boldsymbol{x}_0) = \sqrt{-\frac{a}{2b}} \begin{pmatrix} 2a + 4b(-\frac{a}{2b}) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{14}$$

$$H(\boldsymbol{x}_0) = \begin{pmatrix} 2a + 8b\frac{-a}{2b} + 4b\frac{-a}{2b} & 0 \\ 0 & 2a + 4b\frac{-a}{2b} \end{pmatrix} = \begin{pmatrix} -4a & 0 \\ 0 & 0 \end{pmatrix} \tag{15}$$

This gives the approximate potential

$$V(x,y) \approx -\frac{a^2}{4b} - 2a(x - \sqrt{\frac{-a}{2b}})^2 \tag{16}$$

Combining this last equation with the polar coordinates

$$x = \sqrt{-\frac{a}{2b}} + r\cos(\theta)$$
$$y = r\sin(\theta)$$

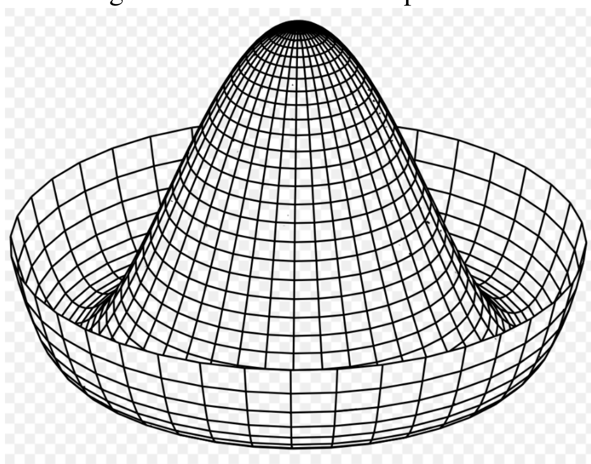
Gives

$$V(r,\theta) = -\frac{a^2}{4b} - 2ar^2\cos^2(\theta) \tag{17}$$

As $a < 0$ and $\cos^2(\theta) \geq 0$, the r^2 term is always non-negative. However, for $\theta = \frac{\pi}{2}$, it vanishes! If you draw a picture, you will see something like figure 2. From the figure, it is clear that $\theta = \frac{\pi}{2}$ corresponds to moving along the trough and show therefore that the ground states are unstable. From the particle point of view, it would seem that the original symmetry is lost due to the θ-dependence of equation (17).

f) If you have done all the above exercises correctly, you will have observed that the ground state in *c* is stable and retains the original symmetry of the potential, whereas the ground state(s) in *d* does not. At what point in your calculation in *d* do you actually lose the symmetry of the initial problem?

Figure 2: The Mexican hat potential.



SOLUTION: It is important to realise that it was our *choice* of ground state $(\sqrt{-\frac{a}{2b}}, 0)$ that broke the symmetry. What is the physical justification for choosing a state like this? The answer is that nature would do the same! The origin is no longer a stable extrema of the potential, and the instability would make any perturbation from $(0,0)$ push the system towards one of the true minimizers (down in the trough). The universe still posses the original rotational symmetry, but it is only reflected in the collection of ground states as a whole and not in any particular one (all points on a circle corresponded to a ground state).

Phenomena like these, where the original rotational symmetry is lost in each of the ground states, but is retained if you look at the collection of ground states, are called *Spontaneous Symmetry Breaking*.

Problem 3

A full treatment of the Higgs mechanism in the Standard model requires writing down a formidable Lagrangian taking into account interactions between many different fields. This exercise is instead meant to show the basic mechanism by which mass terms can be modified in a Lagrangian when symmetries are broken.

Consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - V(x), V(x) = -ax^2 + bx^4 \tag{18}$$

where $a, b > 0$.

a) Complete the square in the potential term and discard any constants.

SOLUTION:

$$V(x) = b(x^4 - \frac{a}{b}x^2) = b(x^2 - \frac{a}{2b})^2 - \frac{a^2}{4b} \simeq b(x^2 - \frac{a}{2b})^2 \equiv by^2 \tag{19}$$

b) Introduce the variable $y = x^2 - \frac{a}{2b}$ and write $\mathcal{L}$ as a function of y.

SOLUTION:

$$\dot{y} = 2x\dot{x} \rightarrow \dot{x} = \frac{\dot{y}}{2x} = \frac{\dot{y}}{2\sqrt{y + \frac{a}{2b}}} \tag{20}$$

giving

$$\mathcal{L}(y) = \frac{1}{2}m\frac{\dot{y}^2}{\frac{2a}{b} + 4y} - by^2 \tag{21}$$

c) Assume y is small and Taylor expand to leading order. What is the mass term in this new Lagrangian?

SOLUTION: The only length scale we have to compare y to is $\frac{a}{2b}$ thus y being small means $y/\frac{a}{2b} \ll 1$. This gives

$$\mathcal{L}(y) \approx \frac{1}{2}m\dot{y}^2(\frac{2a}{b} - 4y) - by^2 = \frac{1}{2}\frac{2ma}{2}\dot{y}^2 - \tilde{V}(y, \dot{y}), \tilde{V}(y, \dot{y}) = by^2 + 2m\dot{y}^2 y \tag{22}$$

We observe that the mass term has been modified and that there are new interactions (see the last term in the modified potential).