NTNU

Institutt for fysikk

FY3403 Particle physics
Problemset 9

SUGGESTED SOLUTION

Problem 8.1

(a) Referring to the figure on page 277,

$$\begin{aligned}
&\int [\bar{v}(2)(ig_e\gamma^\mu)u(1)]\,\frac{-ig_{\mu\nu}}{q^2}\,[\bar{u}(3)(-ig_eQ\gamma^\nu)v(4)] \\
&\qquad \times (2\pi)^4\delta^4(p_1+p_2-q)(2\pi)^4\delta^4(q-p_3-p_4)\,\frac{d^4q}{(2\pi)^4} \\
&= -i\frac{Qg_e^2}{(p_1+p_2)^2}\,[\bar{v}(2)\gamma^\mu u(1)]\,[\bar{u}(3)\gamma_\mu)v(4)]\,.
\end{aligned}$$

$$\mathcal{M} = \frac{Qg_e^2}{(p_1+p_2)^2}\,[\bar{v}(2)\gamma^\mu u(1)]\,[\bar{u}(3)\gamma_\mu)v(4)]\,. \quad \checkmark$$

(b) Using Casimir's trick,

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{1}{4}\sum_{\text{spins}}\left[\frac{Qg_e^2}{(p_1+p_2)^2}\right]^2 \\
&\quad \times [\bar{v}(2)\gamma^\mu u(1)]\,[\bar{u}(3)\gamma_\mu v(4)]\,[\bar{v}(4)\gamma_\nu u(3)]\,[\bar{u}(1)\gamma^\nu v(2)] \\
&= \frac{1}{4}\sum_{\text{spins}}\left[\frac{Qg_e^2}{(p_1+p_2)^2}\right]^2 \\
&\quad \times \mathrm{Tr}\,[\gamma^\mu(\not{p}_1+mc)\gamma^\nu(\not{p}_2-mc)]\,\mathrm{Tr}\,[\gamma_\mu(\not{p}_4-Mc)\gamma_\nu(\not{p}_3+Mc)]\,. \checkmark
\end{aligned}$$

(c) Dropping traces of odd products of gamma matrices,

$$\begin{aligned}
&\mathrm{Tr}\,[\gamma^\mu(\not{p}_1+mc)\gamma^\nu(\not{p}_2-mc)] = \mathrm{Tr}\,(\gamma^\mu\not{p}_1\gamma^\nu\not{p}_2) - (mc)^2\mathrm{Tr}\,(\gamma^\mu\gamma^\nu) \\
&\quad = p_{1\kappa}p_{2\lambda}\mathrm{Tr}\left(\gamma^\mu\gamma^\kappa\gamma^\nu\gamma^\lambda\right) - (mc)^2\mathrm{Tr}\,(\gamma^\mu\gamma^\nu) \\
&\quad = p_{1\kappa}p_{2\lambda}4\left(g^{\mu\kappa}g^{\nu\lambda} - g^{\mu\nu}g^{\kappa\lambda} + g^{\mu\lambda}g^{\nu\kappa}\right) - (mc)^2 4g^{\mu\nu} \\
&\quad = 4\left[p_1^\mu p_2^\nu - g^{\mu\nu}(p_1\cdot p_2) + p_1^\nu p_2^\mu - g^{\mu\nu}(mc)^2\right].
\end{aligned}$$

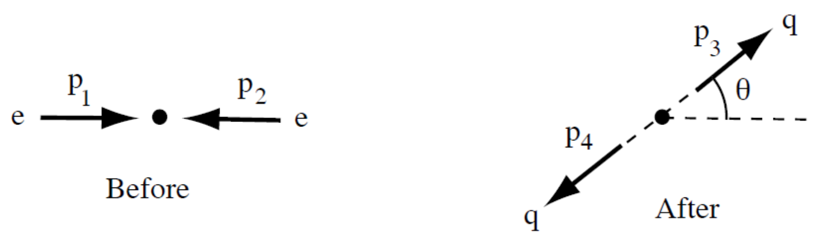
Likewise,

$$\begin{aligned}&\mathrm{Tr}\left[\gamma_\mu(\not p_4 - Mc)\gamma_\nu(\not p_3 + Mc)\right]\\&\quad = 4\left\{p_{4\mu}p_{3\nu} + p_{4\nu}p_{3\mu} - g_{\mu\nu}\left[(p_3\cdot p_4) + (Mc)^2\right]\right\}.\end{aligned}$$

So

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle &= \frac{1}{4}\left[\frac{Qg_e^2}{(p_1+p_2)^2}\right]^2 4\left\{p_1{}^\mu p_2{}^\nu + p_2{}^\nu p_1{}^\mu - g^{\mu\nu}\left[(p_1\cdot p_2) + (mc)^2\right]\right\}\\&\qquad\times 4\left\{p_{3\mu}p_{4\nu} + p_{4\mu}p_{3\nu} - g_{\mu\nu}\left[(p_3\cdot p_4) + (Mc)^2\right]\right\}\\&= 4\left[\frac{Qg_e^2}{(p_1+p_2)^2}\right]^2\left[2(p_1\cdot p_3)(p_2\cdot p_4) + 2(p_1\cdot p_4)(p_2\cdot p_3)\right.\\&\quad -2\,(p_1\cdot p_2)(p_3\cdot p_4) - 2(p_1\cdot p_2)(Mc)^2 - 2(p_1\cdot p_2)(p_3\cdot p_4)\\&\quad -2\,(p_3\cdot p_4)(mc)^2 + 4(p_1\cdot p_2)(p_3\cdot p_4) + 4(p_1\cdot p_2)(Mc)^2\\&\quad\left. + 4(p_3\cdot p_4)(mc)^2 + 4(mMc^2)^2\right]\\&= 8\left[\frac{Qg_e^2}{(p_1+p_2)^2}\right]^2\left[(p_1\cdot p_3)(p_2\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3)\right.\\&\quad\left. +(p_1\cdot p_2)(Mc)^2 + (p_3\cdot p_4)(mc)^2 + 2(mMc^2)^2\right]. \quad \checkmark\end{aligned}$$

(d)



$$p_1 = \left(\frac{E}{c}, \mathbf{p}\right), \quad p_2 = \left(\frac{E}{c}, -\mathbf{p}\right), \quad p_3 = \left(\frac{E}{c}, \mathbf{p}'\right), \quad p_4 = \left(\frac{E}{c}, -\mathbf{p}'\right);$$

$$(p_1+p_2)^2 = \left(\frac{2E}{c}\right)^2, \quad p_1\cdot p_2 = \left(\frac{E}{c}\right)^2 + \mathbf{p}^2, \quad p_3\cdot p_4 = \left(\frac{E}{c}\right)^2 + \mathbf{p}'^2;$$

$$p_1\cdot p_3 = p_2\cdot p_4 = \left(\frac{E}{c}\right)^2 - \mathbf{p}\cdot\mathbf{p}', \quad p_1\cdot p_4 = p_2\cdot p_3 = \left(\frac{E}{c}\right)^2 + \mathbf{p}\cdot\mathbf{p}'.$$

$$\mathbf{p}^2 = \left(\frac{E}{c}\right)^2 - (mc)^2, \quad \mathbf{p}'^2 = \left(\frac{E}{c}\right)^2 - (Mc)^2, \quad \mathbf{p}\cdot\mathbf{p}' = |\mathbf{p}||\mathbf{p}'|\cos\theta.$$

$$p_1\cdot p_2 = 2\left(\frac{E}{c}\right)^2 - (mc)^2, \quad p_3\cdot p_4 = 2\left(\frac{E}{c}\right)^2 - (Mc)^2;$$

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$$(p_1 \cdot p_3)(p_2 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) = 2\left[\left(\frac{E}{c}\right)^4 + (\mathbf{p} \cdot \mathbf{p}')^2\right]$$
$$= 2\left\{\left(\frac{E}{c}\right)^4 + \left[\left(\frac{E}{c}\right)^2 - (mc)^2\right]\left[\left(\frac{E}{c}\right)^2 - (Mc)^2\right]\cos^2\theta\right\}$$

$$\langle|\mathcal{M}|^2\rangle$$
$$= 8\left[\frac{Qg_e^2}{4(E/c)^2}\right]^2\left(2\left\{\left(\frac{E}{c}\right)^4 + \left[\left(\frac{E}{c}\right)^2 - (mc)^2\right]\left[\left(\frac{E}{c}\right)^2 - (Mc)^2\right]\cos^2\theta\right\}\right.$$
$$\left. + \left[2\left(\frac{E}{c}\right)^2 - (mc)^2\right](Mc)^2 + \left[2\left(\frac{E}{c}\right)^2 - (Mc)^2\right](mc)^2 + 2(mMc^2)^2\right)$$
$$= \left[\frac{Qg_e^2}{(E/c)^2}\right]^2\left\{\left(\frac{E}{c}\right)^4 + (ME)^2 + (mE)^2\right.$$
$$\left. + \left[\left(\frac{E}{c}\right)^2 - (mc)^2\right]\left[\left(\frac{E}{c}\right)^2 - (Mc)^2\right]\cos^2\theta\right\}$$
$$= Q^2g_e^4\left\{1 + \left(\frac{mc^2}{E}\right)^2 + \left(\frac{Mc^2}{E}\right)^2\right.$$
$$\left. + \left[1 - \left(\frac{mc^2}{E}\right)^2\right]\left[1 - \left(\frac{Mc^2}{E}\right)^2\right]\cos^2\theta\right\}. \quad \checkmark$$

Problem 8.2

From Eq. 6.47 and Problem 8.1:

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2\frac{(Qg_e^2)^2}{(2E)^2}\left\{1 + \left(\frac{mc^2}{E}\right)^2 + \left(\frac{Mc^2}{E}\right)^2\right.$$
$$\left.\left[1 - \left(\frac{mc^2}{E}\right)^2\right]\left[1 - \left(\frac{Mc^2}{E}\right)^2\right]\cos^2\theta\right\}\frac{\sqrt{E^2 - M^2c^4}}{\sqrt{E^2 - m^2c^4}}$$

$$\sigma = \int\frac{d\sigma}{d\Omega}\sin\theta\,d\theta\,d\phi, \quad \int_0^{2\pi}d\phi = 2\pi, \quad \int_0^{\pi}\sin\theta\,d\theta = 2, \quad \int_0^{\pi}\cos^2\theta\sin\theta\,d\theta = \frac{2}{3}.$$

$$\sigma = \left(\frac{\hbar c Q 4\pi\alpha}{16\pi E}\right)^2 (2\pi)\frac{\sqrt{E^2 - M^2c^4}}{\sqrt{E^2 - m^2c^4}}\left\{2\left[1 + \left(\frac{mc^2}{E}\right)^2 + \left(\frac{Mc^2}{E}\right)^2\right]\right.$$
$$\left. + \frac{2}{3}\left[1 - \left(\frac{mc^2}{E}\right)^2\right]\left[1 - \left(\frac{Mc^2}{E}\right)^2\right]\right\}$$
$$= \frac{\pi Q^2}{12}\left(\frac{\hbar c\alpha}{E}\right)^2 \frac{\sqrt{E^2 - M^2c^4}}{\sqrt{E^2 - m^2c^4}}\left[3 + 3\left(\frac{mc^2}{E}\right)^2 + 3\left(\frac{Mc^2}{E}\right)^2\right.$$
$$\left. + 1 - \left(\frac{mc^2}{E}\right)^2 - \left(\frac{Mc^2}{E}\right)^2 + \left(\frac{mc^2}{E}\right)^2\left(\frac{Mc^2}{E}\right)^2\right]$$
$$= \frac{\pi Q^2}{3}\left(\frac{\hbar c\alpha}{E}\right)^2 \frac{\sqrt{E^2 - M^2c^4}}{\sqrt{E^2 - m^2c^4}}\left[1 + \frac{1}{2}\left(\frac{mc^2}{E}\right)^2 + \frac{1}{2}\left(\frac{Mc^2}{E}\right)^2\right.$$
$$\left. + \frac{1}{4}\left(\frac{mc^2}{E}\right)^2\left(\frac{Mc^2}{E}\right)^2\right]$$
$$= \frac{\pi Q^2}{3}\left(\frac{\hbar c\alpha}{E}\right)^2 \frac{\sqrt{1 - (Mc^2/E)^2}}{\sqrt{1 - (mc^2/E)^2}}\left[1 + \frac{1}{2}\left(\frac{Mc^2}{E}\right)^2\right]\left[1 + \frac{1}{2}\left(\frac{mc^2}{E}\right)^2\right]. \checkmark$$

Problem 8.3

There's a second diagram in the elastic case, and this means that the kinematic factors do not cancel , as they do for the muons.