FY3403 Particle physics

Problemset 8

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SUGGESTED SOLUTION

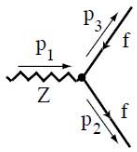
(a) Extra info: to perform the spin summation over outgoing spins (4th equation to 5th equation on the next page), use the result we proved in the lectures that

$$\sum_{\text{spins}} [\bar{u}(a)\Gamma_1 u(b)][\bar{u}(a)\Gamma_2 u(b)]^* = \text{Tr}\{\Gamma_1(\not{p}_b + m_b c)\bar{\Gamma}_2(\not{p}_a + m_a c)\} \tag{1}$$

and then observe that the left-hand side of the above equation is identical to the 4th equation below by using that

$$[\bar{u}(a)\Gamma_2 u(b)]^* = [\bar{u}(a)\Gamma_2 u(b)]^\dagger = [\bar{u}(b)\bar{\Gamma}_2 u(a)]. \tag{2}$$

Finally, use that if $\bar{u}(b), u(b)$ are replaced by $\bar{v}(b), v(b)$ (which is the case in the equations below), the completeness relation for the v-spinors is such that the mass $m_b \to -m_b$ on the right hand side of Eq. (1). This completes the transition from the 4th to 5th equation.



Dropping the superscript f on c_V^f and c_A^f, for the moment,

$$\mathcal{M} = i\epsilon_\mu(1)\bar{u}(2)\left[-\frac{ig_z}{2}\gamma^\mu(c_V - c_A\gamma^5)\right] v(3) \left[\cancel{(2\pi)^4\delta^4(p_1 - p_2 - p_3)}\right];$$

$$|\mathcal{M}|^2 = \left(\frac{g_z}{2}\right)^2 \epsilon_\mu(1)\epsilon_\nu^*(1)\left[\bar{u}(2)\gamma^\mu(c_V - c_A\gamma^5)v(3)\right]\underbrace{\left[\bar{u}(2)\gamma^\nu(c_V - c_A\gamma^5)v(3)\right]^*}_{\bar{v}(3)\gamma^0[\gamma^\nu(c_V - c_A\gamma^5)]^\dagger\gamma^0 u(2)}$$

But

$$\gamma^0\left[\gamma^\nu(c_V - c_A\gamma^5)\right]^\dagger\gamma^0 = \gamma^0(c_V - c_A\gamma^5)^\dagger\gamma^{\nu\dagger}\gamma^0 = \gamma^0(c_V^* - c_A^*\gamma^5)\gamma^{\nu\dagger}\gamma^0$$
$$= (c_V^* + c_A^*\gamma^5)\gamma^0\gamma^{\nu\dagger}\gamma^0 = (c_V^* + c_A^*\gamma^5)\gamma^\nu = \gamma^\nu(c_V^* - c_A^*\gamma^5),$$

so

$$|\mathcal{M}|^2 = \left(\frac{g_z}{2}\right)^2 \epsilon_\mu(1)\epsilon_\nu^*(1)\left[\bar{u}(2)\gamma^\mu(c_V - c_A\gamma^5)v(3)\right]\left[\bar{v}(3)\gamma^\nu(c_V^* - c_A^*\gamma^5)u(2)\right].$$

Summing over the outgoing spins, we get

$$\left(\frac{g_z}{2}\right)^2 \epsilon_\mu(1)\epsilon_\nu^*(1)\mathrm{Tr}\left[\gamma^\mu(c_V - c_A\gamma^5)(\not{p}_3 - m_3c)\gamma^\nu(c_V^* - c_A^*\gamma^5)(\not{p}_2 + m_2c)\right].$$

The trace was calculated in Problem 9.2; quoting Eq. 9.159, with $m_2 = m_3 \to 0$,

$$\left(\frac{g_z}{2}\right)^2 \epsilon_\mu(1)\epsilon_\nu^*(1)4\Big\{\left(|c_V|^2 + |c_A|^2\right)\left[p_2^\mu p_3^\nu + p_2^\nu p_3^\mu - (p_2\cdot p_3)g^{\mu\nu}\right]$$
$$+ i\,(c_Vc_A^* + c_V^*c_A)\,\epsilon^{\mu\nu\lambda\sigma}p_{2_\lambda}p_{3_\sigma}\Big\}.$$

The next step is to average over the (three) spin states of the Z, using the completeness relation in Problem 9.1 (Eq. 9.158):

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{3}g_z^2\left[-g_{\mu\nu} + \frac{p_{1\mu}p_{1\nu}}{(M_Zc)^2}\right](|c_V|^2 + |c_A|^2)\left[p_2^\mu p_3^\nu + p_2^\nu p_3^\mu - g^{\mu\nu}(p_2\cdot p_3)\right].$$

(The $\epsilon^{\mu\nu\lambda\sigma}$ term gives zero, since it is antisymmetric in $\mu \leftrightarrow \nu$, whereas $g_{\mu\nu}$ and $p_{1\mu}p_{1\nu}$ are symmetric.)

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{1}{3}g_Z^2(|c_V|^2+|c_A|^2)\Big\{-2(p_2\cdot p_3)+4(p_2\cdot p_3) \\
&\quad + \frac{1}{(M_Zc)^2}\Big[2(p_1\cdot p_2)(p_1\cdot p_3) - \underbrace{(p_1\cdot p_1)}_{(M_Zc)^2}(p_2\cdot p_3)\Big]\Big\} \\
&= \frac{1}{3}g_z^2(|c_V|^2+|c_A|^2)\left[(p_2\cdot p_3)+2\frac{(p_1\cdot p_2)(p_1\cdot p_3)}{(M_Zc)^2}\right].
\end{aligned}$$

The Golden Rule for 2-body decays in the CM frame (Eq. 6.35) says

$$\begin{aligned}
\Gamma &= \frac{|\mathbf{p}|}{8\pi\hbar M_Z^2 c}\langle|\mathcal{M}|^2\rangle \\
&= \frac{|\mathbf{p}|g_z^2}{24\pi\hbar M_Z^2 c}(|c_V|^2+|c_A|^2)\underbrace{\left[(p_2\cdot p_3)+2\frac{(p_1\cdot p_2)(p_1\cdot p_3)}{(M_zc)^2}\right]}_{\Diamond}.
\end{aligned}$$

In the rest system of the Z,

$$p_1 = (M_Zc,\mathbf{0});\quad p_2 = \left(\frac{M_Zc}{2},\mathbf{p}\right);\quad p_3 = \left(\frac{M_Zc}{2},-\mathbf{p}\right),$$

and (since f is "massless")

$$\left(\frac{M_Zc}{2}\right)^2 - \mathbf{p}^2 = 0;\quad \text{or}\quad \mathbf{p}^2 = \left(\frac{M_Zc}{2}\right)^2.$$

So

$$(p_2\cdot p_3) = \left(\frac{M_Zc}{2}\right)^2 + \mathbf{p}^2 = \frac{(M_Zc)^2}{2};\quad (p_1\cdot p_2) = (p_1\cdot p_3) = \frac{(M_Zc)^2}{2},$$

and hence

$$\Diamond = \frac{(M_Zc)^2}{2} + \cancel{2}\frac{\frac{(M_Zc)^2}{\cancel{2}}\frac{(M_Zc)^2}{2}}{(M_Zc)^2} = (M_Zc)^2.$$

Therefore, (restoring now the superscripts on c_V and c_A)

$$\Gamma = \boxed{\frac{g_z^2(M_Zc^2)}{48\pi\hbar}(|c_V^f|^2+|c_A^f|^2)}.$$

(b) Let $A \equiv (|c_V^f|^2+|c_A^f|^2)$. According to Table 9.1:

$$\begin{array}{lllll}
\nu_e,\nu_\mu,\nu_\tau & : c_V = 0.5; & c_A = 0.5; & A = 0.25+0.25 = 0.5000 \\
e,\mu,\tau & : c_V = -0.0372; & c_A = -0.5; & A = 0.0014+0.25 = 0.2514 \\
u,c,t & : c_V = 0.1915; & c_A = 0.5; & A = 0.0367+0.25 = 0.2867 \\
d,s,b & : c_V = -0.3457; & c_A = -0.5; & A = 0.1195+0.25 = 0.3695.
\end{array}$$

Now

$$\Gamma_{\text{total}} = \frac{g_z^2 M_Z c^2}{48\pi\hbar} \sum_f (|c_V^f|^2 + |c_A^f|^2),$$

where the sum is over the three neutrinos, three charged leptons, u and c (but *not* the t, since it is too heavy: $m_t > M_Z/2$), d, s, and b (and in the case of the quarks, three colors):

$$\sum_f (|c_V^f|^2 + |c_A^f|^2) = 3(0.5000) + 3(0.2514) + 6(0.2867) + 9(0.3695) = 7.2999.$$

The branching ratios ($\Gamma_i/\Gamma_{\text{total}}$) are:

$$\begin{aligned}
&e, \mu, \tau &&: 0.2514/7.2999 = 0.0344, \text{ or } \boxed{3\%} \\
&\nu_e, \nu_\mu, \nu_\tau &&: 0.5000/7.2999 = 0.0685, \text{ or } \boxed{7\%} \\
&u, c &&: 3(0.2867)/7.2999 = 0.1178, \text{ or } \boxed{12\%} \\
&d, s, b &&: 3(0.3695)/7.2999 = 0.1519, \text{ or } \boxed{15\%}
\end{aligned}$$