FY3403 Particle physics
Problemset 4

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SUGGESTED SOLUTION

Problem 1

a) We may rotate the square through angles $\theta = 0, \pi/2, \pi, 3\pi/2$ (elements E, C, C^2, C^3). We may further mirror in the $x-$ and $y-$axes (elements σ_x and σ_y) and in the two diagonals (elements σ_1 and σ_2). Altogether the group has 8 elements.

b)

	$\boldsymbol{E}$	$\boldsymbol{C}$	$\boldsymbol{C^2}$	$\boldsymbol{C^3}$	$\boldsymbol{\sigma_x}$	$\boldsymbol{\sigma_y}$	$\boldsymbol{\sigma_1}$	$\boldsymbol{\sigma_2}$
$\boldsymbol{E}$	E	C	C^2	C^3	σ_x	σ_y	σ_1	σ_2
$\boldsymbol{C}$	C	C^2	C^3	E	σ_1	σ_2	σ_x	σ_y
$\boldsymbol{C^2}$	C^2	C^3	E	C	σ_y	σ_x	σ_2	σ_1
$\boldsymbol{C^3}$	C^3	E	C	C^2	σ_2	σ_1	σ_y	σ_x
$\boldsymbol{\sigma_x}$	σ_x	σ_2	σ_y	σ_1	E	C^2	C^3	C
$\boldsymbol{\sigma_y}$	σ_y	σ_1	σ_x	σ_2	C^2	E	C	C^3
$\boldsymbol{\sigma_1}$	σ_1	σ_x	σ_2	σ_y	C	C^3	E	C^2
$\boldsymbol{\sigma_2}$	σ_2	σ_y	σ_1	σ_x	C^3	C	C^2	E

c) The group is non-abelian. For instance, we find that $\sigma_x \cdot \sigma_1 = C^3$ while $\sigma_1 \cdot \sigma_x = C$.

d) There are quite a few subgroups.

(i) All proper rotations $\{E, C, C^2, C^3\}$.
(ii) Rotations by $0, \pi$, $\{E, C^2\}$.
(iii) Reflections (mirroring) about the x-axis, $\{E, \sigma_x\}$.
(iv) Reflections about the y-axis, $\{E, \sigma_y\}$.
(v) Reflections about the d_1-axis, $\{E, \sigma_1\}$.
(vi) Reflections about the d_2-axis, $\{E, \sigma_2\}$.
(vii) xy-reflections and rotations by π, $\{E, C^2, \sigma_x, \sigma_y\}$.
(viii) $d_1 d_2$-reflections and rotations by π, $\{E, C^2, \sigma_1, \sigma_2\}$.
(ix) Doing nothing, $\{E\}$.

Problem 2

a) If we reverse the direction of the axis we rotate about, and at the same time rotate in the opposite direction, it is still the same physical rotation.

b) We first calculate

$$\begin{aligned}(\hat{n} \cdot \boldsymbol{\sigma})^2 &= (n_x \sigma_X + n_y \sigma_y + n_z \sigma_z)^2 \\ &= n_x^2 + n_y^2 + n_z^2 \end{aligned} \quad (1)$$

because all cross terms between different σ_i-matrices give terms with anticommutators like $\sigma_x\sigma_y + \sigma_y\sigma_x$, and these are zero. Only σ_i^2 terms survive, and these equal 1.

It follows that $(\hat{n}\cdot\boldsymbol{\sigma})^{2n} = 1$ while $(\hat{n}\cdot\boldsymbol{\sigma})^{2n+1} = (\hat{n}\cdot\boldsymbol{\sigma})$. We may then expand the exponential in a Taylor series:

$$\begin{aligned} e^{i\theta\hat{n}\cdot\boldsymbol{\sigma}/2} &= \sum_{m=0}^{\infty} \frac{(i\theta/2)^m}{m!}(\hat{n}\cdot\boldsymbol{\sigma})^m \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{(\theta/2)^{2n}}{2n!} + i\hat{n}\cdot\boldsymbol{\sigma}\sum_{n=0}^{\infty}(-1)^n\frac{(\theta/2)^{2n+1}}{(2n+1)!} \\ &= \cos(\theta/2) + i\sin(\theta/2)\hat{n}\cdot\boldsymbol{\sigma}. \end{aligned} \tag{2}$$

where we in the last equality have used the series expansion of $\cos x$ and $\sin x$ with $x = \theta/2$.

c) With $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, we obtain:

$$\chi_\theta = R(-\theta, \hat{e}_y)\chi_+ = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix} \tag{3}$$

d) Rotating the z unit vector an angle θ about the y-axis gives us $\hat{n} = \sin\theta\hat{e}_x + \cos\theta\hat{e}_z$ so that $\sigma_{\hat{n}} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$. We then find that:

$$\sigma_{\hat{n}}\chi_\theta = \begin{pmatrix} \cos\theta\cos(\theta/2) + \sin\theta\sin(\theta/2) \\ \sin\theta\cos(\theta/2) - \cos\theta\sin(\theta/2) \end{pmatrix} \tag{4}$$

Simplifying this with trigonometric identities provides the desired result.