FY3403 Particle physics
Problemset 11

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SUGGESTED SOLUTION

Problem 1

This exercise is meant to illustrate the important mechanism of *spontaneous symmetry breaking*, where a symmetric potential can lead to ground states where the symmetry is in some sense 'lost'.

Consider a classical particle parameterized by the two-dimensional coordinate $r = (x(t), y(t))$, and let it travel according to the following Hamiltonian

$$H = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - V(x,y) \tag{1}$$

where $V(x,y) = a(x^2+y^2) + b(x^2+y^2)^2 = a(x^2+y^2) + b(x^4+2x^2y^2+y^4)$.

a) What symmetries does the potential possess?

SOLUTION: Rotationally symmetric around the z-axis.

b) Consider the case $a, b > 0$. What is the minimum of the potential? At which point does this minimum occur?

SOLUTION: if a and b are positive, then the potential is nonnegative. This implies $x = y = 0$ is the only minimum of the potential with $V(0,0) = 0$.

c) Suppose a particle is situated at the minimum (x_{min}, y_{min}). A way to check if this minimum is stable is to change to polar coordinates centered at (x_{min}, y_{min}) and calculate how the potential looks like locally around $r = 0$. Taylor expand $V(x,y)$ around (x_{min}, y_{min}) to second order and express the result in polar coordinates. Argue whether the minima are stable or not.

SOLUTION: As we are using polar coordinates centered at the origin, this is just regular polar coordinates. Let $\boldsymbol{x} = (x,y)$ and $\boldsymbol{x}_0 = (x_{min}, y_{min})$

$$V(x,y) \approx V(\boldsymbol{x}_0) + \nabla V(\boldsymbol{x}_0)(\boldsymbol{x}-\boldsymbol{x}_0) + \frac{1}{2}(\boldsymbol{x}-\boldsymbol{x}_0)^T H(\boldsymbol{x}_0)(\boldsymbol{x}-\boldsymbol{x}_0). \tag{2}$$

We thus need the gradient and the hessian of the potential, which are

$$\nabla V = \begin{pmatrix} 2ax + 4bx(x^2+y^2) \\ 2ay + 4by(x^2+y^2) \end{pmatrix}, \; H = \begin{pmatrix} 2a + 8bx^2 + 4b(x^2+y^2) & 8bxy \\ 8bxy & 2a + 8by^2 + 4b(x^2+y^2) \end{pmatrix} \tag{3}$$

which at $(0,0)$ gives

$$\nabla V(0,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \; H(0,0) = \begin{pmatrix} 2a & 0 \\ 0 & 2a \end{pmatrix}. \tag{4}$$

We thus end up with

$$V(x,y) \approx a(x^2+y^2) = ar^2 \tag{5}$$

which, of course, match the potential above. The potential is stable, since any perturbation $r > 0$ would lead to an increase in the energy.

d) Consider now the case $a < 0 < b$. What is the minimal value now? At which point(s) does this minimal value occur?

SOLUTION:

$$V(r) = ar^2 + br^4 = b(r^4 + \frac{a}{b}r^2) = b(r^2 + \frac{a}{2b})^2 - \frac{a^2}{4b} \tag{6}$$

has the minimal value

$$-\frac{a^2}{4b} \tag{7}$$

when

$$r^2 = -\frac{a}{2b}, \tag{8}$$

that is for all values x and y on the circle of radius $-\frac{a}{2b}$.

e) Pick one of the points where the minimal value occur and do the Taylor expansion as above. Argue whether your minima is stable or not. (*Hint:* remember to use polar coordinates centered at the point you chose)

SOLUTION: The calculation of the derivative is the same as above. The only difference is the base point of the polar coordinates, which I choose to be $(\sqrt{-\frac{a}{2b}}, 0)$ on the positive x-axis. From the derivatives above, one obtains

$$\nabla V(\boldsymbol{x}_0) = \sqrt{-\frac{a}{2b}} \begin{pmatrix} 2a + 4b(-\frac{a}{2b}) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{9}$$

$$H(\boldsymbol{x}_0) = \begin{pmatrix} 2a + 8b\frac{-a}{2b} + 4b\frac{-a}{2b} & 0 \\ 0 & 2a + 4b\frac{-a}{2b} \end{pmatrix} = \begin{pmatrix} -4a & 0 \\ 0 & 0 \end{pmatrix} \tag{10}$$

This gives the approximate potential

$$V(x,y) \approx -\frac{a^2}{4b} - 2a(x - \sqrt{\frac{-a}{2b}})^2 \tag{11}$$

Combining this last equation with the polar coordinates

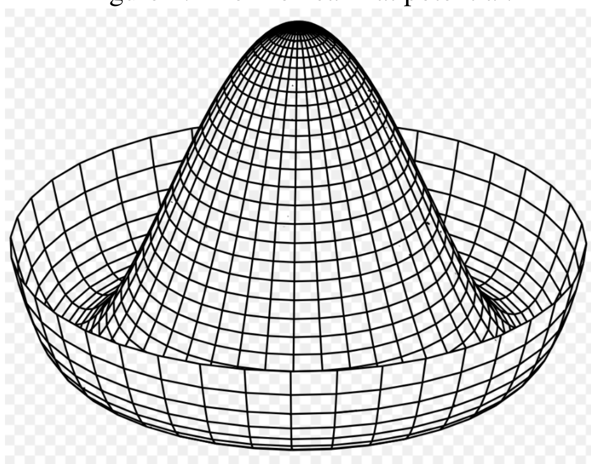
$$x = \sqrt{-\frac{a}{2b}} + r\cos(\theta)$$
$$y = r\sin(\theta)$$

Gives

$$V(r,\theta) = -\frac{a^2}{4b} - 2ar^2\cos^2(\theta) \tag{12}$$

As $a < 0$ and $\cos^2(\theta) \geq 0$, the r^2 term is always non-negative. However, for $\theta = \frac{\pi}{2}$, it vanishes! If you draw a picture, you will see something like figure 1. From the figure, it is clear that $\theta = \frac{\pi}{2}$ corresponds to moving along the trough and show therefore that the ground states are unstable. From the particle point of view, it would seem that the original symmetry is lost due to the θ-dependence of equation (12).

Figure 1: The Mexican hat potential.



f) If you have done all the above exercises correctly, you will have observed that the ground state in *c* is stable and retains the original symmetry of the potential, whereas the ground state(s) in *d* does not. At what point in your calculation in *d* do you actually lose the symmetry of the initial problem?

SOLUTION: It is important to realise that it was our *choice* of ground state $(\sqrt{-\frac{a}{2b}}, 0)$ that broke the symmetry. What is the physical justification for choosing a state like this? The answer is that nature would do the same! The origin is no longer a stable extrema of the potential, and the instability would make any perturbation from $(0,0)$ push the system towards one of the true minimizers (down in the trough). The universe still posses the original rotational symmetry, but it is only reflected in the collection of ground states as a whole and not in any particular one (all points on a circle corresponded to a ground state).

Phenomena like these, where the original rotational symmetry is lost in each of the ground states, but is retained if you look at the collection of ground states, are called *Spontaneous Symmetry Breaking*.

Problem 2

A full treatment of the Higgs mechanism in the Standard model requires writing down a formidable Lagrangian taking into account interactions between many different fields. This exercise is instead meant to show the basic mechanism by which mass terms can be modified in a Lagrangian when symmetries are broken.

Consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - V(x), V(x) = -ax^2 + bx^4 \tag{13}$$

where $a, b > 0$.

a) Complete the square in the potential term and discard any constants.

SOLUTION:

$$V(x) = b(x^4 - \frac{a}{b}x^2) = b(x^2 - \frac{a}{2b})^2 - \frac{a^2}{4b} \simeq b(x^2 - \frac{a}{2b})^2 \equiv by^2 \tag{14}$$

b) Introduce the variable $y = x^2 - \frac{a}{2b}$ and write $\mathcal{L}$ as a function of y.

SOLUTION:

$$\dot{y} = 2x\dot{x} \to \dot{x} = \frac{\dot{y}}{2x} = \frac{\dot{y}}{2\sqrt{y + \frac{a}{2b}}} \tag{15}$$

giving

$$\mathcal{L}(y) = \frac{1}{2}m\frac{\dot{y}^2}{\frac{2a}{b} + 4y} - by^2 \tag{16}$$

c) Assume y is small and Taylor expand to leading order. What is the mass term in this new Lagrangian?

SOLUTION: The only length scale we have to compare y to is $\frac{a}{2b}$ thus y being small means $y/\frac{a}{2b} \ll 1$. This gives

$$\mathcal{L}(y) \approx \frac{1}{2}m\dot{y}^2(\frac{2a}{b} - 4y) - by^2 = \frac{1}{2}\frac{2ma}{2}\dot{y}^2 - \tilde{V}(y,\dot{y}), \tilde{V}(y,\dot{y}) = by^2 + 2m\dot{y}^2 y \tag{17}$$

We observe that the mass term has been modified and that there are new interactions (see the last term in the modified potential).