Inductance

Consider 2 coils of wire

The magnetic field through C1 is

$$\Phi_1 = L_1 I_1 + M_{12} I_2$$

$B_1 \propto I_1$

$B_2 \propto I_2$

and similarly, the flux through C2 is

$$\Phi_2 = L_2 I_2 + M_{21} I_1$$

This is important because

$$\frac{d\Phi_1}{dt} = L_1 \frac{dI_1}{dt} + M_{12}\frac{dI_2}{dt} = -(\mathrm{emf})_1$$

When currents change, that will affect those and other currents by “inductance”.

L_i = self inductance

(how I_i affects itself)

M_{ij} = mutual inductance

(how I_j affects I_i)

<u>Theorem</u> $M_{12} = M_{21}$

<u>Proof</u>

$$\begin{aligned}\Phi_{12} &= M_{12} I_2 = \int_{S_1} \vec{B}_2 \cdot d\vec{a}_1 \\ &= \int_{S_1} \left(\nabla \times \vec{A}_2\right) \cdot d\vec{a}_1 \\ &= \oint_{C_1} \vec{A}_2 \cdot d\vec{\ell}_1\end{aligned}$$

Recall the vector potential,

$$\vec{A}_2(\vec{x}_1) = \oint_{C_2} \frac{\mu_0 I_2 d\vec{\ell}_2}{4\pi \left|\vec{x}_1 - \vec{x}_2\right|}$$

Thus,

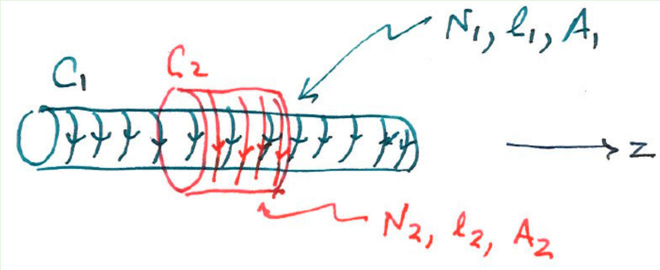
$$M_{12} = \frac{\mu_0}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\vec{\ell}_1 \cdot d\vec{\ell}_2}{\left|\vec{x}_1 - \vec{x}_2\right|}$$

“Neumann’s equation”

$$M_{12} = \frac{\mu_0}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\vec{\ell}_1 \cdot d\vec{\ell}_2}{|\vec{x}_1 - \vec{x}_2|}$$

Q. E. D.

Example. The mutual inductance between coaxial solenoids, Sol_1 and Sol_2, assuming these are densely wound solenoids and Sol_2 is outside Sol_1 ...



Assume $l_2 < l_1$ and $A_2 > A_1$. Neglect end effects.

Case 1 Consider $\Phi_{21} = \int_{S_2} \vec{B}_1 \cdot d\vec{a}_2 = M_{21} I_1$

$\vec{B}_1 = \mu_0 \frac{N_1}{\ell_1} I_1 \hat{k}$ in area A_1

$\Phi_{21} = \mu_0 \frac{N_1}{\ell_1} I_1 N_2 A_1 = M_{21} I_1$

$\therefore M_{21} = \mu_0 N_1 N_2 \frac{A_1}{\ell_1}$ (1)

Case 2 Consider $\Phi_{12} = \int_{S_1} \vec{B}_2 \cdot d\vec{a}_1$

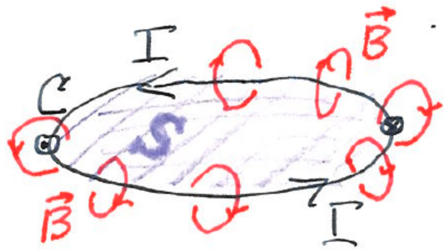
$\vec{B}_2 = \mu_0 \frac{N_2}{\ell_2} I_2 \hat{k}$ in area A_2 and length ℓ_2

$$\Phi_{12} = \left(\mu_0 \frac{N_2}{\ell_2} I_2\right)\underbrace{\left(N_1 \frac{\ell_2}{\ell_1}\right)}_{\text{number of turns of } C_1 \text{ in length } \ell_2} A_1 = M_{12} I_2$$

$\therefore M_{12} = \mu_0 N_1 N_2 A_1/\ell_1$ (2)

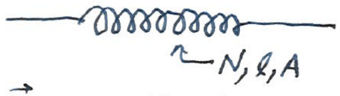
Note that $M_{12} = M_{21}$.

Self Inductance (L)



$$\Phi = \int_S \vec{B}\cdot d\vec{a} = L I$$

Example: a solenoid (= a wire densely wound around a long cylinder)



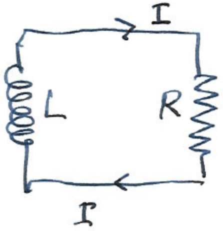
$$\vec{B} = \mu_0 \frac{N}{L} I \hat{k}$$

$$\Phi = \int_S \vec{B}\cdot d\vec{a} = \mu_0 \frac{N}{\ell} I N A$$

$$L = \mu_0 N^2 \frac{A}{\ell}$$

Units: Tm/A x m^2/m = Tm^2/A = H (henry)

LR Circuit



where the initial current is $I(0) = I_0$

The emf around the inductor is

$$\text{emf} = -\frac{d\Phi}{dt} = -\frac{d}{dt}(LI)$$

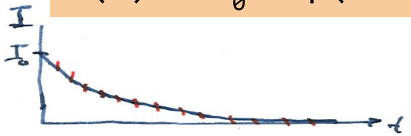
$$= -L\frac{dI}{dt}$$

This emf drives current

$$I = \frac{\text{emf}}{R} \quad \text{(Ohm's law)}$$

(just like a battery!). Thus

```
I = emf /R = - d/dt (LI)    /R
dI /dt   = - (R/L)  I
I(t) = I0 exp(-Rt/L)
```



Energy density of a magnetic field

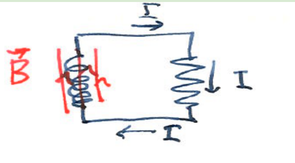
Apply energy conservation to the LR circuit. The power dissipated in resistance is P = I^2 R (Joule's law).

Conservation of energy implies

P dt = dQ V
P = IV = I^2R

$$\frac{dU}{dt} = -P$$

where U is the energy of the magnetic field in the inductor. So



$$\frac{dU}{dt} = -RI_0^2 e^{-2Rt/L}$$

$$U - U_0 = \int_0^t \left(-RI_0^2\right) e^{-2Rt/L}\, dt$$

$$= -RI_0^2 \left(\frac{-L}{2R}\right)\left(e^{-2Rt/L} - 1\right)$$

$$= \frac{1}{2} L I^2 - \frac{1}{2} L I_0^2$$

Evidently, $U = \frac{1}{2} L I^2 = \frac{\Phi^2}{2L}$.

Recall L and Φ for a solenoid....

$1/L$ $\quad$ Φ

$$U = \frac{1}{2}\left(\frac{\ell}{\mu_0 N^2 A}\right)(NBA)^2$$

$$U = \frac{1}{2}\frac{B^2}{\mu_0} A\ell = \frac{1}{2}\frac{B^2}{\mu_0} \times \text{Volume}$$

The *energy density* is $u = \frac{U}{V} = \frac{B^2}{2\mu_0}$.