CLASSICAL MECHANICS TFY4345 - Solution Exercise Set 6

(1a) The force is given by

$$f(r) = -k/r^2 + \beta/r^3 \tag{1}$$

with a belonging potential $V(r) = -k/r + \beta/(2r^2)$. We derived in the lectures that

$$\theta = \int \frac{dr/(r^2)}{\sqrt{2mE/l^2 - 2mV/l^2 - 1/r^2}} + \text{constant} \tag{2}$$

Now insert the $V(r)$ and also introduce $u = 1/r$ to obtain:

$$\theta = \int \frac{du}{\sqrt{2mE/l^2 - 2mku/l^2 - \gamma^2 u^2}} \tag{3}$$

We've here assumed initial conditions so that the integration constant vanishes and defined $\gamma^2 = 1 + \beta m/l^2$. This integral can be looked up in a collection of mathematical formulae (*e.g.* Rottman) and gives us the solution:

$$\theta = -\gamma^{-1} \arccos\left(\frac{p/r - 1}{\varepsilon}\right) \tag{4}$$

where we have defined

$$p = \gamma^2 l^2/(mk), \varepsilon = \sqrt{1 + 2E\gamma^2 l^2/(mk^2)}. \tag{5}$$

We then get the equation for the orbit:

$$r = \frac{p}{1 + \varepsilon\cos(\gamma\theta)} \tag{6}$$

Assume $E < 0$ in which case this equation describes a slowly precessing ellipse. The major halfaxis is $a = p/(1-\varepsilon^2)$ which by direct insertion gives $a = k/(2|E|)$ just like for $\gamma = 1$, i.e. it is unaffected by β.

(1b) Using spherical coordinates, the Lagrange-function reads:

$$L = T - V = l^2 m(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)/2 + mgl\cos\theta \tag{7}$$

The Lagrange-equation for θ gives us:

$$\ddot{\theta} - \sin(2\theta)\dot{\phi}^2/2 + (g/l)\sin\theta = 0 \tag{8}$$

whereas the equation for ϕ provides us with

$$p_\phi = ml^2\sin^2\theta\dot{\phi} = \text{constant} \tag{9}$$

since ϕ is a cyclic coordinate in the Lagrangian. The total energy E is conserved. This may be expressed as

$$E = ml^2\dot{\theta}^2 + \frac{p_\phi^2}{2ml^2\sin^2\theta} - mgl\cos\theta \tag{10}$$

upon eliminating $\dot{\phi}$ in favor of p_ϕ. We thus obtain an effective potential in the following form:

$$E = \frac{1}{2}ml^2\dot{\theta}^2 + V_{\text{eff}}(\theta), \tag{11}$$

where $V_{\text{eff}}(\theta) = \frac{p_\phi^2}{2ml^2\sin^2\theta} - mgl\cos\theta$. This means that $\dot{\theta}^2 = 2(E - V_{\text{eff}})/(ml^2)$, which in turn can be separated to yield:

$$t = \int dt = \sqrt{ml^2/2}\int \frac{d\theta}{\sqrt{E - V_{\text{eff}}(\theta)}}. \tag{12}$$

We also have from the equation for the canonical momentum associated with ϕ (namely p_ϕ) that:

$$\phi = \frac{p_\phi}{\sqrt{2ml^2}}\int \frac{d\theta}{\sin^2\theta\sqrt{E - V_{\text{eff}}(\theta)}} \tag{13}$$

If $p_\phi = 0$, then ϕ is a constant which corresponds to a planar pendulum.