CLASSICAL MECHANICS TFY4345 - Solution Exercise Set 2

(1a) L' and L are equivalent if F satisfies:

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}}\frac{d}{dt}F(q,t) - \frac{\partial}{\partial q}\frac{d}{dt}F(q,t) = 0. \tag{1}$$

Now, we know that

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q}\dot{q}. \tag{2}$$

Inserting this into the first equation, we obtain:

$$\frac{\partial^2 F}{\partial q \partial t} + \frac{\partial^2 F}{\partial q^2}\dot{q} - \frac{\partial^2 F}{\partial q \partial t} - \frac{\partial^2 F}{\partial q^2}\dot{q} = 0. \tag{3}$$

(1b) We have that:

$$\begin{aligned}
[\nabla \times (\nabla \times \mathbf{A})]_i &= \varepsilon_{ijk}\partial_j(\nabla \times \mathbf{A})_k \\
&= \varepsilon_{ijk}\partial_j\varepsilon_{klm}\partial_l A_m \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_j\partial_l A_m \\
&= \partial_i(\nabla \cdot \mathbf{A}) - \nabla^2 A_i.
\end{aligned} \tag{4}$$

We have then shown that $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$.

Exactly the same approach as above for the other cases.

(1c) Frictional force $F_f = -\partial \mathcal{F}/\partial v$. The work performed by the system against friction, per unit time, is $\dot{W} = -F_f v$ which with $\mathcal{F} = Cv^2$ becomes $\dot{W} = 2\mathcal{F}$. The Lagrange equations read:

$$\frac{d}{dt}\frac{\partial L}{\partial v} - \frac{\partial L}{\partial x} + \frac{\partial \mathcal{F}}{\partial v} = 0. \tag{5}$$

Insert the Lagrangian $L = T - V = mv^2/2 - kx^2/2$ and $\mathcal{F} = 3\pi\mu a v^2$ to obtain:

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = 0 \tag{6}$$

where $\lambda = 3\pi\mu a/m$ and $\omega_0 = \sqrt{k/m}$. Assuming $\lambda/\omega_0 \ll 1$, the solution is:

$$x(t) = x_0 e^{-\lambda t}\cos\omega_0 t \tag{7}$$

and the average energy dissipation $\bar{\dot{W}}$ over a period $2\pi/\omega_0$ can be computed by treating $e^{-\lambda t}$ as a constant since it remains virtually unchanged over a time-interval $2\pi/\omega$:

$$\bar{\dot{W}} \simeq m\lambda(\omega_0 x_0)^2 e^{-2\lambda t}. \tag{8}$$