CLASSICAL MECHANICS TFY4345 - Solution Exercise Set 12

(1a) We have $[u,v]_{Q,P} = \partial_Q u \partial_P v - \partial_P u \partial_Q v$. Using that $q = \sqrt{2P/(m\omega)}\sin Q$ and $p = m\omega q \cot Q$, we obtain:

$$\partial_Q u = \partial_q u \partial_Q q + \partial_p u \partial_Q p = \partial_q \sqrt{2P/(m\omega)}\cos Q - \partial_p u \frac{m\omega q}{\sin^2 Q} \tag{1}$$

and also

$$\partial_P v = \partial_q v \partial_P q + \partial_p v \partial_P p = \partial_q v \frac{\sin Q}{\sqrt{2m\omega P}}. \tag{2}$$

In exactly the same way, one can obtain expressions for $\partial_Q v$ and $\partial_P u$. With these 4 quantities in hand, we can now evaluate:

$$[u,v]_{Q,P} = \left(\partial_q u \partial_p v - \partial_p u \partial_q v\right)\frac{m\omega q \sin Q}{\sin^2 Q\sqrt{2m\omega P}} \tag{3}$$

where the term inside the parantheses is seen to be $[u,v]_{q,p}$. The factor that appears afterwards is equal to 1, as seen when inserting for q. Thus, we have shown that

$$[u,v]_{q,p} = [u,v]_{Q,P} \tag{4}$$

for the harmonic oscillator.

(1b) In general, $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ with $A_\mu = (\mathbf{A}, i\phi/c)$. The full matrix form of both $F_{\mu\nu}$ and $L_{\mu\nu}$ is written in the compendium. Since F transforms as

$$F'_{\mu\nu} = L_{\mu\alpha}L_{\nu\beta}F_{\alpha\beta}, \tag{5}$$

it follows that

$$\begin{aligned} E'_1 &= \gamma(E_1 - vB_2), \\ E'_2 &= \gamma(E_2 + vB_1), \\ E'_3 &= E_3. \end{aligned} \tag{6}$$

With $\gamma \simeq 1$, we then have $\mathbf{E}' = \mathbf{E} + \mathbf{v}\times\mathbf{B}$. Writing down the same type of equations for the components B_i of the magnetic field and then taking the limit $\gamma \simeq 1$, provides $\mathbf{B}' = \mathbf{B} - (\mathbf{v}\times\mathbf{E})/c^2$.

(1c) In the instantaneous rest-system we have $\mathbf{j} = \sigma\mathbf{E}$. Since the current density is finite, we must have $\mathbf{E}' \to 0$ when $\sigma \to \infty$. This means that $0 = \mathbf{E} + \mathbf{u}\times\mathbf{B}$, where the terms on the right-hand side are the fields in the lab-system. It follows that $\mathbf{E} = -\mathbf{u}\times\mathbf{B}$ inside the fluid in the lab-system. To lowest order, we may set $\mathbf{B} = \mathbf{B}_0 + O(u)$. See figure in Norwegian solution.