CLASSICAL MECHANICS TFY4345 - Solution Exercise Set 10

(1a) *See figure in the Norwegian version of the solution.* In the rotating coordinate system, the centrifugal force acting on an element of length dr in a distance r from the center equal to $r\omega^2\rho \cdot dr$. Here, ρ is the mass of the rod per unit length. The gravitational pull on the same element dr is $GM\rho \cdot dr/r^2 = g_0 R^2 \rho \cdot dr/r^2$ where g_0 is the gravitational acceleration at the surface of the Earth. Balancing the total graviational and centrifugal force in order to obtain an equilibrium situation, then provides us with:

$$\int_R^{R+L} r\omega^2\rho \cdot dr = g_0 R^2 \rho \int_R^{R+L} dr/r^2. \quad (1)$$

Performing the integration results in:

$$L^2 + 3RL + (2R^2 - 2g_0 R/\omega^2) = 0 \quad (2)$$

This is a 2nd order equation for L whose positive solution reads:

$$L = -3R/2 + \sqrt{R^2 + 8g_0 R/\omega^2}/2 \quad (3)$$

Putting in numbers $R = 6.4$ km, $\omega = 2\pi/(1 \text{ day})$, and $g_0 = 9.8$ m/s^2 gives $L = 1.5 \times 10^5$ km (about halfway to the moon).

(1b) Let $x'y'z' \to 123$. From the relations derived in the compendium/lecture:

$$\begin{aligned}
\omega_1 &= \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi, \\
\omega_2 &= \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi, \\
\omega_3 &= \dot{\phi}\cos\theta + \dot{\psi}, \quad (4)
\end{aligned}$$

we obtain that

$$T = I_1(\omega_1^2 + \omega_2^2)/2 + I_3\omega_3^2/2 \quad (5)$$

is the total energy content as there is no gravity potential, in effect $L = T = E$. Since ϕ and ψ are cyclic coordinates for our Lagrangian, we have that the corresponding canonical momenta are constants:

$$\begin{aligned}
p_\phi &= \partial_{\dot{\phi}} L = (I_1 \sin^2\theta + I_3\cos^2\theta)\dot{\phi} + I_3\dot{\psi}\cos\theta \equiv L_z, \\
p_\psi &= \partial_{\dot{\psi}} L = I_3(\dot{\psi} + \dot{\phi}\cos\theta) \equiv L_3. \quad (6)
\end{aligned}$$

From these equations, we can eliminate the time derivatives of ϕ and ψ in favor of their corresponding canonical momenta (which are constant and renamed L_z and L_3). Inserting this into the energy, we finally arrive at:

$$E = \frac{1}{2}I_1\dot{\theta}^2 + \frac{(L_z - L_3\cos\theta)^2}{2I_1\sin^2\theta} + \frac{L_3^2}{2I_3}. \quad (7)$$