CLASSICAL MECHANICS TFY4345 - Exercise 2

(1a) Show by direct substitution that

$$L'(q,\dot{q},t) = L(q,\dot{q},t) + \frac{dF(q,t)}{dt} \quad (1)$$

where F is an arbitrary function leads to the same Lagrangian equation as $L(q,\dot{q},t)$.

(1b) Use the Levi-Civita tensor to prove the following vector-relations:

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{A}) &= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}, \\ \mathbf{v} \times (\nabla \times \mathbf{v}) &= \frac{1}{2}\nabla v^2 - (\mathbf{v} \cdot \nabla)\mathbf{v}, \\ \nabla \cdot (\mathbf{A} \times \mathbf{B}) &= \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) \end{aligned} \quad (2)$$

where $v = |\mathbf{v}|$.

(1c) A particle with mass m moves with low velocity $\dot{x} = v$. The frictional force is $F_f = -\partial \mathcal{F}/\partial v$, where $\mathcal{F}$ is Rayleigh's dissipation function. Show that if $\mathcal{F} \propto v^2$, the viscous energy loss $\dot{W}_f$ per unit time can be written as $\dot{W}_f = 2\mathcal{F}$. Assume that the particle is a damped oscillator with centre in the origin. The spring constant is k. Assume also that $\mathcal{F} = 3\pi\mu a v^2$ where μ is the dynamic viscosity and a is the particle radius. Start from Lagrange's equation and show that the equation of motion can be written as:

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = 0. \quad (3)$$

Express λ and ω_0 in terms of the above constants. Solve the equation for $x(t)$ assuming that $\dot{x}(0) = 0$ when $\lambda/\omega_0 \ll 1$ and show that one approximately has

$$\overline{\dot{W}_f} = m\lambda(\omega_0 x_0)^2 \mathrm{e}^{-2\lambda t}. \quad (4)$$

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